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CHIRAL PERTURBATION THEORY: EXPANSIONS IN THE MASS OF THE STRANGE QUARK*)

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A B S T R A C T

In a previous paper, we have shown how to systematically determine the low energy structure of the Green's functions in QCD. The present article extends this framework to expansions in the mass of the strange quark. We construct the generating functional of $U(3)\times U(3)$ which allows us to calculate all Green's functions up to and including terms of order p^4 (at fixed radio m/p^2) in terms of a few coupling constants which chiral symmetry leaves undetermined. We calculate the first non-leading term in the quark mass expansion of the order parameters $\langle 0|\bar{u}u|0\rangle$, $\langle 0|\bar{d}d|0\rangle$, $\langle 0|\bar{d}d|0\rangle$, and of the masses and decay constants in the pseudoscalar octet. The three coupling constants which are not fixed by experimental low energy information are estimated by invoking large N_c arguments.

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1. Introduction

In (Gasser and Leutwyler 1983) we have shown how to systematically determine the low energy structure of the Green's functions in QCD. The method is based on a simultaneous expansion in powers of the external momenta and of the quark masses. We have applied the method to an expansion in powers of m_u and m_d at fixed m_s, m_c,... (chiral SU(2)xSU(2)) and have shown that the observed low energy structure of $\pi\pi$ scattering offers a precision test of the theory. This confirms that m_u and m_d are indeed small parameters: if one expands physical quantities like F_π, M_π, form factors or scattering amplitudes in powers of m_u and m_d and only retains the first two terms in this expansion one obtains a very accurate representation of the quantity in question.

In the present paper we extend this framework to expansions in powers of m_s. The approximate validity of SU(3) flavour symmetry indicates that the mass difference $m_s - m_u$ (which is responsible for SU(3) breaking) is small in comparison with the scale of the theory. Since m_u is small, this implies that m_s must also be small; an expansion which treats m_u, m_d and m_s as perturbations should therefore converge rapidly. This hypothesis plays a central role in the determination of the quark mass ratios $m_u : m_d : m_s$ from experimental information about the spectrum of the low lying states (for reviews see Pagels 1975; Gasser and Leutwyler 1982). The purpose of the present paper is to carry the quark mass expansion in powers of m_u, m_d and m_s beyond leading order and to show how to calculate the first nonleading contributions in a systematic manner.

In the first part of this paper (sections 2 - 8) we construct the generating functional of U(3)xU(3) which allows us to calculate all Green's functions to next to leading order in terms of a few effective coupling constants which chiral symmetry leaves undetermined. In sections 9 and 10 we calculate the first nonleading term in the quark mass expansion of the order parameter $\langle 0|\bar{q}q|0\rangle$ and of the masses and coupling constants in the pseudoscalar octet. We then discuss the role of the η' in chiral perturbation theory. The considerations on large N_c and on the Zweig rule in section 13 allow us to estimate those of the effective coupling constants which are not fixed directly by experimental low energy information.

2. Chiral symmetry

The Green's functions associated with the vector, axial vector, scalar and pseudoscalar quark currents and with the operator $G_{\mu\nu}\tilde{G}^{\mu\nu}$ are generated by the functional

$$\exp iZ = \langle 0\ \mathrm{out} \mid 0\ \mathrm{in} \rangle_{v,a,s,p,\theta} \tag{2.1}$$

where $\langle 0_{out} \mid 0_{in} \rangle$ is the vacuum-to-vacuum transition amplitude in the presence of external fields, determined by the Lagrangian

$$\mathcal{L} = \mathcal{L}^0_{QCD} + \bar{q}\gamma^\mu\{v_\mu(x) + \gamma_5 a_\mu(x)\}q - \bar{q}\{s(x) - i\gamma_5 p(x)\}q - \frac{1}{16\pi^2}\theta(x)\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu} \tag{2.2}$$

L^0_{QCD} is what remains of the QCD Lagrangian if the masses of the three light quarks and the vacuum angle are set equal to zero. The external fields $v_\mu(x)$, $a_\mu(x)$, $s(x)$, $p(x)$ are hermitean 3x3 matrices in flavour space. (We have included the QCD coupling constant in the definition of the gluon field strength: $G_{\mu\nu} = i[D_\mu, D_\nu]$. The symbol tr_c denotes the trace over colour indices.) Note that the mass matrix

$$m = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \tag{2.3}$$

of the three light quarks is contained in the scalar field $s(x)$, whereas the mass terms of the heavy quarks are retained in L^0_{QCD}.

If one expands the generating functional $Z(v,a,s,p,\theta)$ around $v_\mu = a_\mu = s = p = 0$, $\theta(x) = \theta_0$ one obtains the Green's functions of QCD with massless u, d and s quarks. The Green's functions of the real world are obtained by expanding Z around $v_\mu = a_\mu = p = 0$, $s(x) = M$, $\theta(x) = \theta_0$.

Formally, the vacuum-to-vacuum amplitude is invariant with respect to local U(3)xU(3) transformations:

$$q(x) \rightarrow V_R(x)\tfrac{1}{2}(1+\gamma_5)q(x) + V_L(x)\tfrac{1}{2}(1-\gamma_5)q(x) \tag{2.4}$$

which induce a gauge transformation of the external fields

$$
\begin{aligned}
v'_\mu + a'_\mu &= V_R (v_\mu + a_\mu) V_R^+ + i V_R \partial_\mu V_R^+ \\
v'_\mu - a'_\mu &= V_L (v_\mu - a_\mu) V_L^+ + i V_L \partial_\mu V_L^+ \\
s' + i p' &= V_R (s + i p) V_L^+
\end{aligned}
\tag{2.5}
$$

For an infinitesimal chiral transformation

$$
\begin{aligned}
V_R(x) &= 1 + i\alpha(x) + i\beta(x) + \dots \\
V_L(x) &= 1 + i\alpha(x) - i\beta(x) + \dots
\end{aligned}
\tag{2.6}
$$

the change in the external fields is given by

$$
\begin{aligned}
\delta v_\mu &= \partial_\mu \alpha + i[\alpha, v_\mu] + i[\beta, a_\mu] \\
\delta a_\mu &= \partial_\mu \beta + i[\alpha, a_\mu] + i[\beta, v_\mu] \\
\delta s &= i[\alpha, s] - \{\beta, p\} \\
\delta p &= i[\alpha, p] + \{\beta, s\}
\end{aligned}
\tag{2.7}
$$

The anomalies of the fermion determinant however break chiral invariance - the generating functional is not invariant under the transformations (2.5). The change in Z may be given explicitly, provided one simultaneously transforms the external field θ(x) (Bardeen 1969; Wess and Zumino 1971; Fujikawa 1980):

$$
\begin{aligned}
\delta\Theta(x) &= -2 \,\mathrm{tr}_f\, \beta(x) \\
\delta Z &= -\int dx \,\mathrm{tr}_f \{\beta(x)\Omega(x)\} \\
\Omega(x) &= \frac{N_c}{16\pi^2} \varepsilon^{\alpha\beta\mu\nu} \Big[v_{\alpha\beta} v_{\mu\nu} + \tfrac{4}{3} \nabla_\alpha a_\beta \nabla_\mu a_\nu + \tfrac{2i}{3} \{ v_{\alpha\beta}, a_\mu a_\nu \} \\
&\quad + \tfrac{8i}{3} a_\mu v_{\alpha\beta} a_\nu + \tfrac{4}{3} a_\alpha a_\beta a_\mu a_\nu \Big]
\end{aligned}
\tag{2.8}
$$

$$v_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha - i\left[v_\alpha, v_\beta\right]$$
$$\nabla_\alpha a_\beta = \partial_\alpha a_\beta - i\left[v_\alpha, a_\beta\right] \tag{2.8}$$

(N_c is the number of colours and tr_f denotes the trace operation in flavour space. In the following we will often drop the index f, since the traces occurring evidently concern flavour matrices.) The transformation law (2.8) of the generating functional embodies all Ward identities associated with U(3)xU(3) (for a review of the properties of the generating functional, see Crewther 1980).

In (2.8) the action of U(3)xU(3) on the angle $\theta(x)$ is only specified for infinitesimal transformations. Since the group U(3)xU(3) is not simply connected one may arrive at the same global transformation by performing sequences of infinitesimal steps which cannot be continuously deformed into one another. Consider for simplicity only the Green's functions of the scalar and pseudoscalar currents and of the operator $G_{\mu\nu}\tilde{G}^{\mu\nu}$, i.e. put $v_\mu = a_\mu = 0$ (only space independent chiral transformations preserve this restriction). Since the contribution from the anomaly vanishes in this case ($\Omega = 0$) we have

$$Z(0,0,s',p',\theta') = Z(0,0,s,p,\theta) \tag{2.9}$$

The transformed fields $s'(x)$, $p'(x)$ are given in (2.5); the value of θ' depends on the sequence of infinitesimal steps used to reach the transformation V_R, V_L from the origin. Consider e.g. the transformation $V_R = \exp(-i\gamma)$, $V_L = \mathbb{1}$ where γ is a diagonal matrix with only one nonvanishing eigenvalue. If we let this eigenvalue increase from zero to 2π, the matrix V_R makes an excursion from the origin back to the origin. Since in this example we have $\alpha = \beta = -\frac{1}{2}\gamma$, the transformed vacuum angle is given by $\theta' = \theta + \mathrm{tr}\,\gamma$. As the eigenvalue of γ increases from 0 to 2π the vacuum angle changes by 2π, whereas s and p return to their original values. The transformation law (2.9) therefore requires Z to be periodic in $\theta(x)$ with period 2π:

$$Z(0,0,s,p,\theta+2\pi) = Z(0,0,s,p,\theta) \tag{2.10}$$

The scalar and pseudoscalar Green's functions of QCD, obtained by expanding Z around the point $s(x) = M$, $p(x) = 0$, $\theta(x) = \theta_0$ are therefore periodic functions of θ_0. In particular, the spectrum of the theory belonging to the mass matrix and to the vacuum angle θ_0 is the same as the spectrum associated with M,

$\theta_0 + 2\pi$. (In the above discussion we have implicitly assumed that the ground state of the theory does not undergo discontinuous changes under the global transformations considered - see section 4 for some comments concerning this problem.)

We add a remark about the form of the quark mass matrix. The quark masses originate in the asymmetries of the electroweak vacuum. Since the electroweak interactions do not conserve parity there is no a priori reason for the quark mass term of QCD to be parity invariant. With a suitable U(3)xU(3) transformation of the quark fields the general mass term $\bar{q}(s_0 - i\gamma_5 p_0)q$ may however always be brought to the form $\bar{q}Mq$ where M is diagonal with real positive eigenvalues m_u, m_d, m_s (Weinberg 1973)

$$V_R (s_0 + i p_0) V_L^+ = m \qquad (2.11)$$

If the determinant of $s_0 + ip_0$ is not real and positive, this transformation contains a chiral U(1) rotation (tr $\beta \neq 0$). In order for the theory to remain the same one therefore has to transform the vacuum angle accordingly. If the theory is originally characterized by the mass matrix $s(x) = s_0$, $p(x) = p_0$ and by the vacuum angle $\theta(x) = \theta_0$, the equivalent set of parameters is

$$s(x) = m, \; p(x) = 0, \; \Theta(x) = \bar{\Theta} \qquad (2.12)$$

where

$$\bar{\Theta} = \Theta_0 + \arg \det (s_0 + i p_0) \qquad (2.13)$$

is the chirally invariant vacuum angle. The spectrum of the theory therefore depends on s_0, p_0 and θ_0 only through the positive eigenvalues of the quark mass matrix and through the chirally invariant angle $\bar{\theta}$. We use a quark field basis in which the mass matrix is diagonal and positive. In this basis the Green's functions of QCD are obtained by expanding the generating functional Z around the point $v_\mu = a_\mu = p = 0$, $s(x) = M$, $\theta(x) = \bar{\theta}$.

If $\bar{\theta}$ is not a multiple of π the theory does not conserve parity. From the experimental upper bound on the electric dipole moment of the neutron one concludes (Baluni 1979; Crewther, Di Vecchia, Veneziano and Witten 1979; a less stringent limit is obtained from $\eta \to 2\pi$, see Shifman, Vainshtein and Zakharov 1980) that $\bar{\theta}$ must be very close to a multiple of π. Since the theory is periodic in $\bar{\theta}$ with period 2π we are faced with two distinct possibilities: $\bar{\theta} \simeq 0$ and $\bar{\theta} \simeq \pi$.

In the chiral limit ($m_u = m_d = m_s = 0$) the nine vector currents are exactly conserved. We assume that the vacuum respects this symmetry such that the spectrum of the massless theory consists of degenerate multiplets of SU(3). In the chiral limit the 8 axial currents are also exactly conserved. We assume that the ground state spontaneously breaks this symmetry with a nonvanishing order parameter $<0|\bar{q}q|0>$ as a quantitative measure of the vacuum asymmetry. The spectrum of the massless theory then contains 8 pseudoscalar Goldstone bosons which in the chiral limit are exactly massless. (In the real world these particles are not massless, because the quark mass matrix produces explicit breaking of chiral symmetry. The squares of the masses of π, K and η are proportional to $m_u + m_d$, $m_u + m_s$ and $\frac{1}{3}(m_u + m_d + 4 m_s)$ respectively.)

The divergence of the ninth axial current remains different from zero if the quark masses are turned off. As pointed out by Witten (1979), the anomaly responsible for the nonvanishing divergence is however of order $1/N_c$. In the limit $N_c \to \infty$ the theory does contain a ninth massless pseudoscalar, the η'. For large N_c the square of the η' mass is of order Λ^2/N_c. In most of the following we do not treat N_c as large, but stick to the physical value $N_c = 3$. The η' does then not play any special role in the analysis of the low energy structure of the theory. (The effect of the η' on the low energy properties of the Green's functions and the large N_c limit are discussed in detail in sections 12 and 13).

3. Effective low energy Lagrangian

The behaviour of the Green's functions at small momenta is reflected in the structure of the generating functional for external fields which vary slowly in comparison with the scale of the theory. An expansion of the Green's functions in powers of the momenta corresponds to an expansion of the generating functional in powers of the derivatives of the external fields. The low energy expansion is not a simple Taylor series, however: the Goldstone bosons associated with the spontaneously broken symmetry generate poles at small momenta, e.g. at $p^2 = M_\pi^2 = O(\mathcal{M})$. The Green's functions admit a Taylor series expansion in the momenta only if $p^2 << M_\pi^2$. To describe their behaviour for values of p^2 of the order of M_π^2 or larger we keep the ratio M_π^2/p^2 fixed, i.e. treat both p and $\mathcal{M}$ as small in comparison with the scale of the theory, but allow the ratio $\mathcal{M}/p^2$ to have any value. A method which allows one to systematically carry out the corresponding expansion of the generating functional is described in detail in II where we restricted ourselves to an expansion in m_u and m_d at fixed m_s. In the following we assume that the reader is familiar with the external field technology and only indicate the modifications needed to extend the framework from SU(2)xSU(2) to U(3)xU(3). A detailed account of the method and references to the literature on the subject may be found in II.

Since the spontaneous breakdown of SU(3)xSU(3) to SU(3) gives rise to 8 Goldstone bosons, the effective low energy Lagrangian involves 8 Goldstone fields rather than three as in the case of SU(2)xSU(2). We collect these fields in the unitary 3x3 matrix U(x):

$$U(x)\,U(x)^{+} = \mathbb{1} \tag{3.1}$$

which under chiral U(3)xU(3) transforms according to the linear representation

$$U'(x) = V_R(x)\,U(x)\,V_L^{+}(x) \tag{3.2}$$

A unitary 3x3 matrix contains nine degrees of freedom rather than eight. To eliminate the ninth field we impose a condition on the determinant of U(x). (Alternatively, one may retain this field to describe the degrees of freedom associated with the η' meson. We will discuss the role of the η' in sections 12 and 13 in connection with the large N_c limit. For $N_c = 3$ the mass $M_{\eta'}$ is of the order of the scale of the theory. At energies small in comparison with this scale, the region we are analyzing here, the η' degrees of freedom are frozen; the pole factors $(M^2_{\eta'} - p^2)^{-1}$ associated with the propagation of an η' may be expanded as $M^{-2}_{\eta'}\{1 + p^2/M^2_{\eta'} + \ldots\}$. In the low energy expansion the presence of the η' only shows up indirectly, in the same manner as any other bound state whose mass remains different from zero in the chiral limit, through a contribution to the expansion coefficients.)

The standard constraint det U = 1 is not consistent with the transformation law (3.2), because det $V_R V_L^{+}$ differs from one for chiral U(1) rotations. We instead put

$$\det U(x) = e^{-i\Theta(x)} \tag{3.3}$$

which is consistent with (3.2).

In accordance with (3.2) the covariant derivative of U(x) is defined as

$$\nabla_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu) \tag{3.4}$$

The effective Lagrangian is a function of the meson field U(x) and its derivatives as well as of the external fields and their derivatives. It is convenient to write this function in the form

$$\mathcal{L} = \mathcal{L}(U, \theta; v_\mu, a_\mu, \nabla_\mu U, \nabla_\mu \theta; s, p, \partial_\mu v_\nu, \dots) \tag{3.5}$$

where we have introduced the covariant derivative of $\theta(x)$ as

$$\nabla_\mu \theta = \partial_\mu \theta + 2\,\mathrm{tr}\, a_\mu \tag{3.6}$$

and have ordered the fields according to their low energy dimension (U, θ count as fields of order 1; v_μ, a_μ, $\partial_\mu U$, $\partial_\mu \theta$ as order p; s(x), p(x) as order p^2 etc.)
To leading order in the low energy expansion the generating functional coincides with the classical action

$$Z = \int dx\, \mathcal{L} \tag{3.7}$$

The transformation law (2.8) states that Z is gauge invariant up to a contribution from the anomalies which is of order p^4. The contributions to the generating functional of order 1 and of order p^2 must therefore be gauge invariant. Since all other variables in (3.5) are of order p or higher, the general effective Lagrangian of order 1 is a function of U and θ only. In order for this function to be invariant under chiral transformations it can depend on U(x) only through det U. The constraint (3.3) fixes det U in terms of $\theta(x)$; at order 1 the Lagrangian is therefore independent of the meson field. Finally, since the external field $\theta(x)$ transforms in a nontrivial manner under chiral transformations, the Lagrangian cannot depend on $\theta(x)$ either. The most general chirally invariant effective Lagrangian of order 1 is therefore an irrelevant constant: the low energy expansion of Z starts with contributions of order p^2 (chiral symmetry implies derivative coupling).

To determine the general effective Lagrangian at order p^2 we observe that gauge invariance permits the fields v_μ, a_μ only to enter through the covariant derivative ∇_μ and through the field strength tensors $F^R_{\mu\nu}$, $F^L_{\mu\nu}$ defined by

$$F^I_{\mu\nu} = \partial_\mu F^I_\nu - \partial_\nu F^I_\mu - i\left[F^I_\mu, F^I_\nu\right] \quad ; I = R, L$$

$$F^R_\mu = v_\mu + a_\mu \tag{3.8}$$

$$F^L_\mu = v_\mu - a_\mu$$

Up to and including terms of order p^2 the Lagrangian therefore only contains the variables U, θ, $\nabla_\mu U$, $\nabla_\mu \theta$, $\nabla_\mu \nabla_\nu U$, $\partial_\mu \nabla_\nu \theta$, s, p, $F^R_{\mu\nu}$, $F^L_{\mu\nu}$. Terms linear in $F^R_{\mu\nu}$, $F^L_{\mu\nu}$ are forbidden by Lorentz invariance and terms containing the second derivatives $\nabla_\mu \nabla_\nu U$, $\partial_\mu \nabla_\nu \theta$ may be eliminated by partial integration. Using the identity

$$\mathrm{tr}\left(U^+ \nabla_\mu U\right) = -i \nabla_\mu \Theta \tag{3.9}$$

which follows from (3.3), one easily demonstrates that to order p^2 the general effective Lagrangian consistent with Lorentz invariance and with chiral symmetry is of the form

$$\mathcal{L}_1 = \frac{F_0^2}{4}\left[\mathrm{tr}\left\{\nabla_\mu U^+ \nabla^\mu U\right\} + 2B_0\, \mathrm{tr}\left\{(s-ip)U\right\} + 2B_0^*\, \mathrm{tr}\left\{(s+ip)U^+\right\}\right]$$
$$+ \frac{H_0}{12} \nabla_\mu \Theta \nabla^\mu \Theta \tag{3.10}$$

where F_0, B_0 and H_0 are arbitrary constants; F_0 and H_0 are real, B_0 may be complex.

4. Ground state

To obtain the generating functional at leading order in the low energy expansion it suffices to evaluate the classical action associated with the Lagrangian (3.10); the field $U(x)$ occurring in this Lagrangian is determined by the external fields through the equations of motion

$$U \nabla^\mu \nabla_\mu U^+ - \nabla_\mu \nabla^\mu U U^+ - 2B_0 U (s-ip) + 2B_0^* (s+ip) U^+ = i\lambda \mathbb{1} \tag{4.1}$$

(λ is a Lagrange multiplier which stems from the constraint (3.3)). We first consider the ground state of the system, i.e. analyze the equations of motion in the absence of external perturbations:

$$v_\mu = a_\mu = p = 0 \; , \; \theta(x) = \bar{\theta} \; , \; s(x) = \mathcal{M} \tag{4.2}$$

(we discuss the properties of the ground state for arbitrary values of $\bar{\theta}$; as mentioned in section 2 the physical value of $\bar{\theta}$ is either $\bar{\theta} \simeq 0$ or $\bar{\theta} \simeq \pi$).

The equations of motion state that U(x) is to be evaluated at a local extremum of the classical action. In fact, in order for the solution to be stable with respect to infinitesimal perturbations the extremum must be a local minimum of the corresponding Euclidean action. (The spectrum of the fluctuations around a local maximum exhibits imaginary masses: fluctuations are not restored, but instead explode (Dashen 1971).)

In the absence of external fields, the minimum of the Euclidean action occurs for a constant field $U(x) = U_0$, characterized by the condition

$$\mathrm{tr}\left\{ \mathcal{M} \left(B_0 U_0 + B_0^* U_0^+ \right) \right\} = \text{maximum} \tag{4.3}$$

subject to the constraints

$$U_0 U_0^+ = \mathbb{1} \quad ; \quad \det U_0 = e^{-i\bar{\theta}} \tag{4.4}$$

The solution U_0 in particular determines the vacuum expectation values of the scalar and pseudoscalar operators:

$$\langle 0 | \bar{q}_L \lambda q_R | 0 \rangle = -\frac{1}{2} F_0^2 B_0 \, \mathrm{tr} \left(\lambda U_0 \right) \tag{4.5}$$

(This relation is easily obtained by calculating the response of the generating functional to an infinitesimal change in the external fields s(x) and p(x).) The equations of motion require

$$B_0 U_0 \mathcal{M} - B_0^* \mathcal{M} U_0^+ = -\frac{i}{2} \lambda \mathbb{1} \tag{4.6}$$

Taking the commutator of this relation with the matrix MU_0^+ one obtains

$$\left[U_0 \mathcal{M} , \mathcal{M} U_0^+ \right] = 0$$

or, equivalently,

$$[U_0, m^2] = 0 \tag{4.7}$$

If the three quark masses m_u, m_d and m_s are different, The matrix U_0 must therefore be diagonal:

$$U_0 = \begin{pmatrix} e^{i\varphi_u} & & \\ & e^{i\varphi_d} & \\ & & e^{i\varphi_s} \end{pmatrix} ; \quad \varphi_u + \varphi_d + \varphi_s = -\bar{\Theta} \tag{4.8}$$

(Often in the literature (e.g. Witten 1980) a different convention is used: $U_0 \to U_0^{-1}$, $\phi_q \to -\phi_q$.) The values of the angles ϕ_u, ϕ_d and ϕ_s at which the minimum occurs depend on the phase of the constant B_0. To pin down this phase we invoke the assumption that the ground state of QCD does not break parity spontaneously: we assume that for the physical value of $\bar{\theta}$ (idealized to 0 or π) the ground state is an eigenstate of parity. Accordingly, the expectation value of the pseudoscalar operators $\bar{q}i\gamma_5\lambda q$ must vanish. The relation (4.5) shows that this is the case only if

$$B_0 U_0 = B_0^* U_0^+ \tag{4.9}$$

which, in view of det $U_0 = \pm 1$ implies that B_0^3 is real, i.e. that the phase of B_0 must be a multiple of $\pi/3$. Since one may always replace the field $U(x)$ by $U(x)$ exp $2\pi i/3$ without leaving the constraints (3.1) and (3.3) we only need to distinguish two cases:

I. B_0 real and positive

II. B_0 real and negative

In this phase convention the condition (4.9) states that the diagonal matrix U_0 is real and the equations of motion take the form (Dashen 1971; Nuyts 1971; Witten 1980; Crewther 1980)

$$\begin{aligned} m_u \sin\varphi_u &= m_d \sin\varphi_d = m_s \sin\varphi_s \\ \varphi_u + \varphi_d + \varphi_s &= -\bar{\Theta} \end{aligned} \tag{4.10}$$

In principle, these equations determine the three angles as functions of $\bar{\theta}$. Actually, for each value of $\bar{\theta}$ there are several inequivalent solutions; we are interested in the particular solution which realizes the minimum of the Euclidean action. It suffices to discuss the properties of this solution for $B_0 > 0$: denoting the matrix U_0 which realizes the minimum for $B_0 > 0$ by $U_0(\bar{\theta})$ the minimum for a negative value of B_0 occurs at $-U_0(\bar{\theta}+\pi)$. Suppose, therefore, that B_0 is positive. If $\bar{\theta}$ vanishes, the minimum occurs at $U_0 = \mathbb{1}$, $\phi_u = \phi_d = \phi_s = 0$. As $\bar{\theta}$ increases the constraint $\phi_u + \phi_d + \phi_s = -\bar{\theta}$ drives the angles ϕ_q away from the origin. We refer the reader to Crewther (1980) for an analysis of the properties of the ground state in the general case. For the physical values of the quark masses which satisfy $0 < m_u < m_d \ll m_s$ the solution is approximately given by

$$e^{i\varphi_u} \simeq e^{-i\frac{\bar{\theta}}{2}} z/|z| \; ; \; e^{i\varphi_d} \simeq e^{-i\frac{\bar{\theta}}{2}} z^*/|z| \; ; \; \varphi_s \simeq 0$$

$$z = m_u e^{i\frac{\bar{\theta}}{2}} + m_d e^{-i\frac{\bar{\theta}}{2}} \tag{4.11}$$

In particular, at $\bar{\theta} = \pi$ the matrix U_0 is of the form $U_0 = \mathrm{diag}\,(-1,1,1)$. As shown by Crewther, Di Vecchia, Veneziano and Witten (1979) the excitations of the ground state $U_0 = \mathrm{diag}\,(-1,1,1)$ have a mass spectrum which is at variance with observation. If the constant B_0 is positive, then $\bar{\theta} = \pi$ is therefore ruled out. Accordingly, we have two possibilities for the physical ground state: either $\bar{\theta}$ is close to zero and B_0 is positive or $\bar{\theta}$ is close to π and B_0 is negative. In both cases the matrix U_0 describing the ground state is proportional to the unit matrix. As far as the effective Lagrangian is concerned the two cases are equivalent, because what counts is the product B_0U_0 which is a positive multiple of the unit matrix in both cases.

Before we proceed we emphasize that the constants which occur in the effective Lagrangian are not basic constants of nature, but are induced constants reflecting properties of the ground state of the theory. It is not established that the same effective Lagrangian describes the low energy structure of QCD for all values of the ratio $m_u : m_d : m_s$ and of the vacuum angle $\bar{\theta}$. The Ward identities of chiral symmetry which determine the structure of the effective Lagrangian are infinitesimal constraints valid as long as the ground state of the theory responds smoothly to changes in the parameters of QCD and in the external fields. It is well-known that for some range of these parameters the matrix U_0 changes discontinuously as the vacuum angle $\bar{\theta}$ passes through π: If m_u is taken larger than the reduced mass of d and s, the ground state U_0 approaches a complex matrix as $\bar{\theta} = \pi$ is approached from below, signalling spontaneous breakdown of parity. If $\bar{\theta}$ crosses π the ground state flips from U_0 to U_0^* and then again varies continuously as $\bar{\theta}$ is increased to 2π where the ground state returns smoothly to $U_0 = \mathbb{1}$. It is however not clear that the ground state of QCD does behave in this manner if one varies $\bar{\theta}$ all the way from 0 to 2π. An alternative possiblity is that the ground state changes discontinuously already if $\bar{\theta}$ reaches $\pi/2$. If this is the case the regions $-\frac{\pi}{2} < \bar{\theta} < \frac{\pi}{2}$ and $\frac{\pi}{2} < \bar{\theta} < \frac{3\pi}{2}$ are described by two different effective Lagrangians which differ in the sign of B_0 (the sign of B_0 is determined by the ground state of the system: the sign may be the one for which the ground state energy is lower.)

The point here is that if the ground state undergoes discontinuous changes, we see no reason for the effective Lagrangian not to notice - unfortunately, this implies that the effective Lagrangian framework is not good enough to determine the values of $\bar{\theta}$ at which discontinuities in the structure of the ground state occur. (Indeed one can construct an effective Lagrangian including the η' for which the ground state jumps at $\bar{\theta} = \frac{\pi}{2}$, just as well as one can construct one for which discontinuities occur only at $\bar{\theta} = \pi$.)

In the following we consider the low energy expansion of the Green's functions for the physical values of the parameters in the QCD Lagrangian. In this context the problem just discussed is of no relevance: what counts is that the ground state U_0 is proportional to the unit matrix. For simplicity we disregard the small parity violation permitted by the experimental bounds on the dipole moment of the neutron and, for definiteness, take $\bar{\theta} = 0$, $B_0 > 0$, $U_0 = \mathbb{1}$.

5. Low energy expansion at leading order

To work out the Green's functions at leading order in the low energy expansion it suffices to determine the classical action associated with the Lagrangian (3.10) as a power series in the external fields. It is convenient to simplify the constraint (3.3) by writing

$$U(x) = \tilde{U}(x) \exp\left\{-\frac{i}{3}\theta(x)\right\} \tag{5.1}$$

such that $\det \tilde{U} = 1$. The matrix $\tilde{U}$ then collects the Goldstone fields in the standard manner. One may e.g. represent $\tilde{U}$ in the form

$$\tilde{U}(x) = \exp(i\varphi) \tag{5.2}$$

where ϕ is hermitean and traceless and may therefore be decomposed as

$$\varphi = \varphi_a \lambda^a \tag{5.3}$$

in terms of eight real fields $\phi_1, \ldots, \phi_8$. The covariant derivative of U then takes the form

$$\nabla_\mu U = \left\{ \nabla_\mu \tilde{U} - \frac{i}{3} \nabla_\mu \Theta\, \tilde{U} \right\} \exp\left\{ -\frac{i}{3} \Theta \right\} \tag{5.4}$$

where $\nabla_\mu \tilde{U}$ only involves the traceless parts of v_μ and a_μ:

$$\begin{aligned} \nabla_\mu \tilde{U} &= \partial_\mu \tilde{U} - i(\tilde{v}_\mu + \tilde{a}_\mu)\tilde{U} + i\tilde{U}(\tilde{v}_\mu - \tilde{a}_\mu) \\ \tilde{v}_\mu &= v_\mu - \frac{1}{3}\,\mathrm{tr}\, v_\mu = v_\mu^a \frac{\lambda^a}{2} \\ \tilde{a}_\mu &= a_\mu - \frac{1}{3}\,\mathrm{tr}\, a_\mu = a_\mu^a \frac{\lambda^a}{2} \end{aligned} \tag{5.5}$$

Using the property

$$\mathrm{tr}\left(\nabla_\mu \tilde{U}\, \tilde{U}^+ \right) = 0 \tag{5.6}$$

which follows from $\det \tilde{U} = 1$, one obtains the following representation of the Lagrangian

$$\begin{aligned} \mathcal{L}_1 &= \frac{F_0^2}{4} \left\{ \mathrm{tr}\, \nabla_\mu \tilde{U}^+ \nabla^\mu \tilde{U} + \mathrm{tr}(\chi \tilde{U}^+ + \chi^+ \tilde{U}) \right\} + \frac{\tilde{H}_0}{12} \nabla_\mu \Theta \nabla^\mu \Theta \\ \chi(x) &= 2B_0 \left\{ s(x) + i p(x) \right\} \exp\left\{ \frac{i}{3} \Theta(x) \right\} \\ \tilde{H}_0 &= H_0 + F_0^2 \end{aligned} \tag{5.7}$$

The leading low energy representation of the generating functional is given by the value of the classical action

$$Z_1 = \int dx\, \mathcal{L}_1 \tag{5.8}$$

evaluated at the solution to the classical equations of motion for $\tilde{U}$

$$\nabla^\mu \nabla_\mu \tilde{U}\, \tilde{U}^+ - \tilde{U}\, \nabla^\mu \nabla_\mu \tilde{U}^+ + \tilde{U} \chi^+ - \chi \tilde{U}^+ - \frac{1}{3}\,\mathrm{tr}(\tilde{U}\chi^+ - \chi \tilde{U}^+) = 0 \tag{5.9}$$

The Lagrangian L_1 does not contain the field tr v_μ. Green's functions involving the singlet vector current $\bar{q}\gamma_\mu q$ do therefore not receive any contribution at leading order in the low energy expansion. The field tr a_μ does occur in L_1, but only through the contact term $\nabla_\mu\theta\nabla^\mu\theta$. At leading order in the low energy expansion the only nonvanishing Green's functions containing the singlet axial current $\bar{q}\gamma_\mu\gamma_5 q$ are therefore the two-point functions

$$
\begin{aligned}
& i\int dx\, e^{ip(x-y)} \langle 0|T A_\mu(x) A_\nu(y)|0\rangle = g_{\mu\nu}\, 6\tilde{H}_0 + O(p^2) \\
& i\int dx\, e^{ip(x-y)} \langle 0|T A_\mu(x)\, \omega(y)|0\rangle = i p_\mu \tilde{H}_0 + O(p^3) \\
& A_\mu = \bar{q}\gamma_\mu\gamma_5 q \\
& \omega = \frac{1}{16\pi^2} \operatorname{tr} G_{\mu\nu}\tilde{G}^{\mu\nu}
\end{aligned}
\tag{5.10}
$$

The representation (5.7) also shows that (apart from the contact term $\nabla_\mu\theta\nabla^\mu\theta$) the external field $\theta(x)$ only occurs together with s and p, in the combination $(s + ip) \exp \frac{1}{3} i\theta$ (this matrix is invariant under chiral U(1)-transformations; the phase of its determinant is the chirally invariant vacuum angle $\bar{\theta}$). A perturbation of the ground state $s = M$, $v = a = p = \theta = 0$ by $\delta\theta$ is therefore equivalent to the perturbation $\delta p = \frac{1}{3} M\delta\theta$. Illustrations of this property will be given below.

The effective Lagrangian L_1 contains three low energy constants F_0, B_0, H_0. The physical significance of these constants is easily established by working out a few Green's functions. The vacuum expectation values of the operators $\bar{u}u$, $\bar{d}d$, $\bar{s}s$ are obtained by calculating the change in the classical action induced by a change in the external field s(x). The result reads

$$
\langle 0|\bar{q}\lambda q|0\rangle = -F_0^2 B_0 \operatorname{tr}\lambda \{1 + O(m)\}
\tag{5.11}
$$

To determine the two-point functions of the axial currents

$$
A_\mu^a = \bar{q}\gamma_\mu\gamma_5 \frac{\lambda^a}{2} q
\tag{5.12}
$$

of the pseudoscalar operators

$$P^a = \bar{q}\, i\gamma_5 \lambda^a q \tag{5.13}$$

and of the winding number density $G\tilde{G}$ we need to solve the equations of motion for $\hat{U}$ to first order in the external fields a_μ, p and θ. Expressed in terms of the field ϕ defined in (5.2) the equations of motion are

$$\Box\varphi + B_0\{m,\varphi\} - \frac{2}{3} B_0 \operatorname{tr}(m\varphi) = 2\partial^\mu \tilde{a}_\mu + 2 B_0\{\tilde{p} + \frac{1}{3}\theta m - \frac{1}{9}\theta \operatorname{tr} m\} \tag{5.14}$$

where $\tilde{p}$, $\tilde{a}_\mu$ denote the traceless parts of p and a_μ.

To diagonalize the mass term we introduce eight traceless 3x3 matrices λ_{π^+}, λ_{π^-}, ..., λ_η with the properties

$$B_0\{m,\lambda_P\} - \frac{2}{3} B_0 \operatorname{tr}(m\lambda_P) = \mathring{M}_P^2 \lambda_P \tag{5.15}$$
$$\operatorname{tr}(\lambda_P \lambda_{P'}^+) = 2\delta_{PP'}$$

Explicitly, using the standard phase conventions (de Alfaro, Fubini, Furlan and Rossetti 1973) the matrices λ_P are given by

$$\begin{aligned}
\lambda_{\pi^+} &= -\frac{1}{\sqrt{2}}(\lambda^1 + i\lambda^2) & \lambda_{\pi^-} &= \frac{1}{\sqrt{2}}(\lambda^1 - i\lambda^2) \\
\lambda_{K^+} &= -\frac{1}{\sqrt{2}}(\lambda^4 + i\lambda^5) & \lambda_{K^-} &= \frac{1}{\sqrt{2}}(\lambda^4 - i\lambda^5) \\
\lambda_{K^0} &= -\frac{1}{\sqrt{2}}(\lambda^6 + i\lambda^7) & \lambda_{\bar{K}^0} &= -\frac{1}{\sqrt{2}}(\lambda^6 - i\lambda^7) \\
\lambda_{\pi^0} &= \cos\varepsilon\, \lambda^3 + \sin\varepsilon\, \lambda^8 & \lambda_\eta &= -\sin\varepsilon\, \lambda^3 + \cos\varepsilon\, \lambda^8
\end{aligned} \tag{5.16}$$

The condition (5.15) fixes the $\pi^0 - \eta$ mixing angle ε

$$\operatorname{tg} 2\varepsilon = \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}} \; ; \; \hat{m} = \frac{1}{2}(m_u + m_d) \tag{5.17}$$

as well as the eigenvalues

$$\overset{\circ}{M}{}^2_{\pi^\pm} = (m_u + m_d) B_0$$
$$\overset{\circ}{M}{}^2_{K^\pm} = (m_u + m_s) B_0$$
$$\overset{\circ}{M}{}^2_{K^0} = \overset{\circ}{M}{}^2_{\bar{K}^0} = (m_d + m_s) B_0 \tag{5.18}$$
$$\overset{\circ}{M}{}^2_{\pi^0} = (m_u + m_d) B_0 - \tfrac{4}{3} (m_s - \hat{m}) B_0 \sin^2\varepsilon / \cos 2\varepsilon$$
$$\overset{\circ}{M}{}^2_{\eta} = \tfrac{2}{3} (\hat{m} + 2m_s) B_0 + \tfrac{4}{3} (m_s - \hat{m}) B_0 \sin^2\varepsilon / \cos 2\varepsilon$$

(The index o in $\overset{\circ}{M}_P$ is to remind us that these formulae for the masses of the pseudoscalar mesons only hold at leading order; the corrections of order M^2 will be given in section 10).

The low energy behaviour of the two-point functions involving the axial vector and pseudoscalar currents and the winding number density $\omega(x)$ may be read off from the expression for the classical action, calculated to second order in the external fields a_μ, p, θ. The low energy representation of $\langle 0|T G\tilde{G}\, G\tilde{G}|0\rangle$, e.g., becomes

$$i \int dx\, e^{ip(x-y)} \langle 0 | T\, \omega(x)\, \omega(y) | 0 \rangle = \sum_{P=\pi^0,\eta} \frac{|\langle 0 | \omega | P \rangle|^2}{\overset{\circ}{M}{}^2_P - p^2}$$
$$- \tfrac{1}{9} B_0 F_0^2 (m_u + m_d + m_s) + \tfrac{1}{6} \tilde{H}_0 p^2 + O(p^4) \tag{5.19}$$
$$\langle 0 | \omega | \pi^0 \rangle = \tfrac{1}{2} (m_d - m_u) B_0 F_0 \left(1 - \tfrac{4}{3} \sin^2\varepsilon\right) / \cos\varepsilon$$
$$\langle 0 | \omega | \eta \rangle = \frac{2}{3\sqrt{3}} (m_s - \hat{m}) B_0 F_0 \left(1 - 4 \sin^2\varepsilon\right) \cos\varepsilon / \cos 2\varepsilon$$

The vacuum-to-meson matrix elements of the operators A^a_μ and P^a may be extracted from the corresponding two-point functions with the result

$$\langle 0|A^a_\mu|P\rangle = i\,p_\mu F_0 \tfrac{1}{2}\,\mathrm{tr}(\lambda^a\lambda_P)\{1+O(m)\}$$
$$\langle 0|P^a|P\rangle = B_0F_0\,\mathrm{tr}(\lambda^a\lambda_P)\{1+O(m)\} \tag{5.20}$$

The expressions for the matrix elements of the operators ω and P^a illustrate the general property of the generating functional pointed out above, which implies

$$\langle 0|\omega|P\rangle = -\frac{1}{6}\sum_a \mathrm{tr}(m\lambda^a)\langle 0|P^a|P\rangle + O(m^2) \tag{5.21}$$

The matrix elements of the singlet operators $\bar{q}\gamma_\mu\gamma_5 q$ and $\bar{q}\gamma_5 q$ do not receive a contribution at leading order

$$\langle 0|\bar{q}\gamma_\mu\gamma_5 q|P\rangle = O(p_\mu m)$$
$$\langle 0|\bar{q}\gamma_5 q|P\rangle = O(m) \tag{5.22}$$

6. Effective Lagrangian to order p^4

At order p^4 the generating functional contains three different classes of contributions:

(i) The anomaly is of order p^4; we therefore need to construct a functional Z_A which has the property that its change under a chiral gauge transformation reproduces the anomaly.

(ii) Once the anomaly is taken care of, we need to determine the most general gauge invariant effective Lagrangian L_2 and add the corresponding action $Z_2 = \int dx L_2$.

(iii) Finally, we have to calculate the one loop graphs associated with the lowest order Lagrangian L_1 - these graphs are also of order p^4.

Together with the lowest order term Z_1 given in the preceding section, the sum of these contributions

$$Z = Z_{1} + Z_{A} + Z_{2} + Z_{\text{one loop}} \tag{6.1}$$

generates the general solution of the Ward identities at first nonleading order in the low energy expansion.

A functional Z_A which does correctly reproduce the anomaly was constructed by Wess and Zumino (1971) (Witten has recently given a remarkable geometric interpretation of this expression in the limit $v_\mu = a_\mu = 0$ (Witten 1983)). The construction goes as follows. Denote the generator of an infinitesimal chiral transformation by $D(\beta)$

$$D(\beta)f(v,a,s,p,\theta) = \int dx \Big[\mathrm{tr}\Big\{ i[\beta, a_\mu] \frac{\delta f}{\delta v_\mu(x)} + \nabla_\mu \beta \frac{\delta f}{\delta a_\mu(x)} - \{\beta, p\} \frac{\delta f}{\delta s(x)} + \{\beta, s\} \frac{\delta f}{\delta p(x)} \Big\} - 2\,\mathrm{tr}\beta \frac{\delta f}{\delta \theta(x)} \Big] \tag{6.2}$$

The condition to be solved, (2.8), then amounts to

$$D(\beta) Z_A = - \int dx\, \mathrm{tr}\{\beta(x)\Omega(x)\}. \tag{6.3}$$

where Ω is an expression involving only v_μ and a_μ, explicitly given in (2.8). The operator exp $D(\beta)$ generates the global transformation

$$V_R(x) = V_L^{+}(x) = \exp i\beta(x) \tag{6.4}$$

Indeed, one easily verifies that the action of this operator on the external fields is given by

$$\begin{aligned}
e^{D(\beta)}\{v_\mu(x) + a_\mu(x)\} &= e^{i\beta(x)}(i\partial_\mu + v_\mu + a_\mu)e^{-i\beta(x)} \\
e^{D(\beta)}\{v_\mu(x) - a_\mu(x)\} &= e^{-i\beta(x)}(i\partial_\mu + v_\mu - a_\mu)e^{i\beta(x)} \\
e^{D(\beta)}\{s(x) + ip(x)\} &= e^{i\beta(x)}\{s(x) + ip(x)\}e^{i\beta(x)} \\
e^{D(\beta)}\theta(x) &= \theta(x) - 2\,\mathrm{tr}\,\beta(x)
\end{aligned} \tag{6.5}$$

The equations of motion for the field U(x) determine this matrix as a functional of the external fields. Under a chiral transformation of the external fields U(x) transforms according to the representation

$$e^{D(\beta)}U(x) = e^{i\beta(x)}U(x)e^{i\beta(x)} \tag{6.6}$$

This property may be used to transform U(x) into the unit matrix; it suffices to choose the matrix $\beta(x)$ such that

$$e^{-2i\beta(x)} = U(x) \tag{6.7}$$

(U(x) is a power series in the external fields, starting with the unit matrix. Eq. (6.7) uniquely specifies $\beta(x)$ as an analogous power series if we require $\beta(x)$ to vanish in the absence of external fields.) The condition (6.3) may be solved with a functional Z_A which depends on the external fields only through v_μ, a_μ and U

$$Z_A = Z_A(v, a, U) \tag{6.8}$$

The condition (6.3) amounts to a differential equation which determines the dependence of this functional on the field U. If we supplement the differential equation with the boundary condition

$$Z_A(v, a, \mathbb{1}) = 0 \tag{6.9}$$

(which is consistent with the invariance of Z_A with respect to the gauge transformations generated by the vector current) then the functional is fixed uniquely. Indeed, if we apply a global chiral transformation specified by the particular matrix $\beta(x)$ which satisfies (6.7) we obtain

$$e^{D(\beta)} Z_A(v, a, U) = 0 \tag{6.10}$$

Using (6.3) this implies the explicit representation

$$Z_A(v, a, U) = -\sum_{n=1}^{\infty} \frac{1}{n!} \int dx \, \mathrm{tr}\left\{\beta(x)\left[D(\beta)\right]^{n-1} \Omega(x)\right\} \tag{6.11}$$

The first term in this series in particular contains the anomaly (Adler 1969; Bell and Jackiw 1969) responsible for the decays $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$:

$$Z_A(v,a,U) = -\frac{N_c}{16\pi^2}\int dx\, \mathrm{tr}\{\beta(x)\varepsilon^{\alpha\beta\mu\nu} v_{\alpha\beta}(x)\, v_{\mu\nu}(x)\} + \dots \tag{6.12}$$

As is well-known, the higher order terms in (6.11) do not vanish even if the vector and axial vector fields are switched off - the anomaly generates interactions among five or more Goldstone bosons. (See (Witten 1983) for an analysis of the structure of Z_A (0,0,U).)

Next, we construct the general gauge invariant Lagrangian of order p^4. Since we do not intend to analyze the Green's functions involving the singlet currents $\bar{q}\gamma_\mu q$, $\bar{q}\gamma_\mu\gamma_5 q$ or the winding number density ω beyond leading order, we disregard the corresponding external fields

$$\mathrm{tr}\, v_\mu = \mathrm{tr}\, a_\mu = \Theta = 0 \tag{6.13}$$

The effective Lagrangian of order p^2 then simplifies to

$$\mathcal{L}_1 = \frac{F_0^2}{4}\{\mathrm{tr}\, \nabla_\mu U^+ \nabla^\mu U + \mathrm{tr}(\chi^+ U + \chi U^+)\} \tag{6.14}$$

and the constraint which eliminates the U(1)-field associated with the η' becomes

$$\det U = 1 \tag{6.15}$$

Since we need the Lagrangian L_2 only at tree graph level, we may use the classical field equations (5.9) obeyed by U to simplify the general expression of order p^4. Using the procedure outlined in section 3 to impose gauge invariance, Lorentz invariance, P, C and T one finds the following expression for the general Lagrangian of order p^4

$$\begin{aligned}\mathcal{L}_2 = {} & L_1 \langle \nabla^\mu U^+ \nabla_\mu U\rangle^2 + L_2 \langle \nabla_\mu U^+ \nabla_\nu U\rangle\langle \nabla^\mu U^+ \nabla^\nu U\rangle \\ & + L_3 \langle \nabla^\mu U^+ \nabla_\mu U \nabla^\nu U^+ \nabla_\nu U\rangle + L_4 \langle \nabla^\mu U^+ \nabla_\mu U\rangle\langle \chi^+ U + \chi U^+\rangle \\ & + L_5 \langle \nabla^\mu U^+ \nabla_\mu U(\chi^+ U + U^+ \chi)\rangle\end{aligned} \tag{6.16}$$

$$
\begin{aligned}
&+ L_6 \langle \chi^+ U + \chi U^+ \rangle^2 + L_7 \langle \chi^+ U - \chi U^+ \rangle^2 \\
&+ L_8 \langle \chi^+ U \chi^+ U + \chi U^+ \chi U^+ \rangle \\
&- i L_9 \langle F^R_{\mu\nu} \nabla^\mu U \nabla^\nu U^+ + F^L_{\mu\nu} \nabla^\mu U^+ \nabla^\nu U \rangle \\
&+ L_{10} \langle U^+ F^R_{\mu\nu} U F^{L\mu\nu} \rangle + H_1 \langle F^R_{\mu\nu} F^{\mu\nu R} + F^L_{\mu\nu} F^{\mu\nu L} \rangle + H_2 \langle \chi^+ \chi \rangle
\end{aligned} \tag{6.16}
$$

where <A> stands for the trace of the matrix A. The field strength tensors $F^R_{\mu\nu}$, $F^L_{\mu\nu}$ are defined in (3.8)

At leading order two constants F_0, B_0 suffice to determine the low energy behaviour of the Green's functions (recall that we disregard the singlet vector and axial currents) - at first nonleading order we need 10 additional low energy coupling constants $L_1, \ldots, L_{10}$. (Although the contact terms H_1, H_2 are of no physical significance, they are needed as counter terms in the renormalization of the one loop graphs.)

7. Loops

To evaluate the one loop graphs generated by the Lagrangian $\mathcal{L}_1$ we consider the neighbourhood of the solution $\bar{U}(x)$ to the classical equations of motion. Denoting the square root of this solution by u(x)

$$\bar{U} = u^2 \tag{7.1}$$

we write the expansion around $\bar{U}$ in the form

$$U = u\left(1 + i\xi - \tfrac{1}{2}\xi^2 + \ldots\right)u \tag{7.2}$$

where $\xi(x)$ is a traceless hermitean matrix. The number of flavours does not play a crucial role in the following anlysis. We perform the one loop calculations for the nonlinear σ-model of SU(N)xSU(N) and put the number N of flavours equal to three only at the end. (The case N = 2 offers a welcome check with the one loop analysis of SU(2)xSU(2) described previously (ref. II).) In the remainder of this section (unless explicitly stated otherwise) the quantities U, ξ etc. are NxN matrices.

Inserting the expansion (7.2) in the expression for the action and retaining terms up to and including ξ^2 one obtains

$$\int dx \mathcal{L}_1 = \overline{Z}_1 + \frac{F_0^2}{4}\int dx\, tr\Big\{\nabla^\mu(u^+\xi u^+)\nabla_\mu(u\xi u) - \frac{1}{2}\nabla^\mu\overline{U}^+\nabla_\mu(u\xi^2 u)$$
$$-\frac{1}{2}\nabla^\mu\overline{U}\,\nabla_\mu(u^+\xi^2 u^+) - \frac{1}{2}\xi^2(u\chi^+ u + u^+\chi u^+)\Big\} \tag{7.3}$$

To simplify this expression we introduce the antihermitean matrices

$$\Gamma_\mu = \frac{1}{2}\left[u^+, \partial_\mu u\right] - \frac{i}{2}u^+ F_\mu^R u - \frac{i}{2} u F_\mu^L u^+$$
$$\Delta_\mu = \frac{1}{2}u^+\nabla_\mu\overline{U}u^+ = -\frac{1}{2}u\nabla_\mu\overline{U}^+ u \tag{7.4}$$

and define the covariant derivative of ξ as

$$d_\mu\xi = \partial_\mu\xi + [\Gamma_\mu, \xi]\,. \tag{7.5}$$

In this notation we have

$$\nabla_\mu(u\xi u) \equiv \partial_\mu(u\xi u) - iF_\mu^R u\xi u + iu\xi u F_\mu^L$$
$$= u(d_\mu\xi + \{\Delta_\mu, \xi\})u \tag{7.6}$$

and the action takes the form

$$\int dx\mathcal{L}_1 = \overline{Z}_1 + \frac{F_0^2}{4}\int dx\, tr\{d_\mu\xi d^\mu\xi - [\Delta_\mu,\xi][\Delta^\mu,\xi] - \xi^2\sigma\}$$
$$\sigma = \frac{1}{2}(u\chi^+ u + u^+\chi u^+) \tag{7.7}$$

More explicitly, in terms of the components ξ^a

$$\xi = \xi^a\lambda^a \quad , \; tr(\lambda^a\lambda^b) = 2\delta^{ab}$$

the quadratic form (7.7) may be written as

$$\int dx\mathcal{L}_1 = \overline{Z}_1 - \frac{F_0^2}{2}(\xi, D\xi) \tag{7.8}$$

where

$$(f,g) = \sum_a \int dx\, f^a(x)\, g^a(x) \tag{7.9}$$

and D stands for the differential operator

$$\begin{aligned} D^{ab}\xi^b &= d^\mu d_\mu \xi^a + \hat{\sigma}^{ab}\xi^b \\ d_\mu \xi^a &= \partial_\mu \xi^a + \hat{\Gamma}_\mu^{ab}\xi^b \\ \hat{\Gamma}_\mu^{ab} &= -\tfrac{1}{2}\,\mathrm{tr}([\lambda^a,\lambda^b]\,\Gamma_\mu) \\ \hat{\sigma}^{ab} &= \tfrac{1}{2}\,\mathrm{tr}([\lambda^a,\Delta_\mu][\lambda^b,\Delta^\mu]) + \tfrac{1}{4}\,\mathrm{tr}(\{\lambda^a,\lambda^b\}\sigma) \end{aligned} \tag{7.10}$$

The contribution of the one loop graphs to the generating functional is given by the Gaussian integral

$$\begin{aligned} e^{iZ_{\text{one loop}}} &= \int d\mu[\xi]\, e^{-\frac{i}{2}F_0^2(\xi, D\xi)} \\ &= N\,(\det D)^{-\frac{1}{2}} \end{aligned} \tag{7.11}$$

To evaluate $Z_{\text{one loop}}$ we therefore need to calculate the determinant of the differential operator D

$$Z_{\text{one loop}} = \frac{i}{2}\ln\det D \tag{7.12}$$

We regularize the determinant by working in d dimensions (see II for details). The ultraviolet divergences produce poles in ln det D at d = 0,2,4,....

$$Z_{\text{one loop}} = \int dx \left\{ -\frac{1}{d}\,\mathrm{Sp}\,\mathbb{1} + \frac{1}{4\pi(d-2)}\,\mathrm{Sp}\,\hat{\sigma} - \frac{1}{(4\pi)^2(d-4)}\,\mathrm{Sp}\left(\frac{1}{12}\hat{\Gamma}_{\mu\nu}\hat{\Gamma}^{\mu\nu} + \frac{1}{2}\hat{\sigma}^2\right) + \dots \right\} \tag{7.13}$$

where Sp denotes the trace in the space of the $(N^2-1)\times(N^2-1)$ matrices $\mathbb{1}$, $\hat{\sigma}$, $\hat{\Gamma}_{\mu\nu}\hat{\Gamma}^{\mu\nu}$. The matrix $\hat{\sigma}^{ab}$ is defined in (7.10) and $\hat{\Gamma}_{\mu\nu}$ stands for the field strength belonging to $\hat{\Gamma}_\mu$

$$\hat{\Gamma}_{\mu\nu} = \partial_\mu\hat{\Gamma}_\nu - \partial_\nu\hat{\Gamma}_\mu + [\hat{\Gamma}_\mu, \hat{\Gamma}_\nu] \tag{7.14}$$

The trace of $\hat{\Gamma}_{\mu\nu}\hat{\Gamma}_{\mu\nu}$ may be worked out by noting that the matrix $\hat{\Gamma}_{\mu\nu}$ is given by

$$\hat{\Gamma}^{ab}_{\mu\nu} = -\frac{1}{2}\mathrm{tr}\left([\lambda^a,\lambda^b]\Gamma_{\mu\nu}\right) \tag{7.15}$$

in terms of the field strength associated with the NxN matrix Γ_μ defined in (7.4). The generators λ^a of SU(N) obey the completeness relation

$$\sum_{a=1}^{N^2-1}\mathrm{tr}(\lambda^a A\lambda^a B) = -\frac{2}{N}\mathrm{tr}(AB) + 2\,\mathrm{tr}A\,\mathrm{tr}B \tag{7.16}$$

from which it follows that

$$\sum_{a=1}^{N^2-1}\mathrm{tr}(\lambda^a A)\,\mathrm{tr}(\lambda^a B) = 2\,\mathrm{tr}(AB) - \frac{2}{N}\mathrm{tr}A\,\mathrm{tr}B \tag{7.17}$$

Using these relations one easily shows that

$$\mathrm{Sp}\,\hat{\Gamma}_{\mu\nu}\hat{\Gamma}^{\mu\nu} = 2N\,\mathrm{tr}\,\Gamma_{\mu\nu}\Gamma^{\mu\nu} \tag{7.18}$$

Analogously, one obtains

$$\begin{aligned}\mathrm{Sp}\,\hat{\sigma}^2 = {} & 4\langle\Delta^\mu\Delta^\nu\rangle\langle\Delta_\mu\Delta_\nu\rangle + 2\langle\Delta^\mu\Delta_\mu\rangle^2 + 2N\langle\Delta^\mu\Delta_\mu\Delta^\nu\Delta_\nu\rangle \\ & - 2N\langle\sigma\Delta^\mu\Delta_\mu\rangle - 2\langle\sigma\rangle\langle\Delta^\mu\Delta_\mu\rangle \\ & + \frac{N^2-4}{2N}\langle\sigma^2\rangle + \frac{N^2+2}{2N^2}\langle\sigma\rangle^2\end{aligned} \tag{7.19}$$

where $\langle A\rangle$ again denotes the trace of A; the matrix σ is defined in (7.7)

Finally, it remains to express the quantities $\Gamma_{\mu\nu}$ and Δ_μ in terms of $\bar{U}$, $\nabla_\mu\bar{U}$ and $F^R_{\mu\nu}$, $F^L_{\mu\nu}$. Calculating $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)(u\xi u)$ with (7.6) one finds

$$\Gamma_{\mu\nu} = -[\Delta_\mu,\Delta_\nu] - \frac{i}{2}u^+F^R_{\mu\nu}u - \frac{i}{2}uF^L_{\mu\nu}u^+ \tag{7.20}$$

Inserting this in (7.18) one obtains

$$\mathrm{Sp}\,\hat{\Gamma}_{\mu\nu}\hat{\Gamma}^{\mu\nu} = \frac{N}{4}\langle\nabla^\mu\bar{U}^+\nabla^\nu\bar{U}\nabla_\mu\bar{U}^+\nabla_\nu\bar{U}\rangle - \frac{N}{4}\langle\nabla^\mu\bar{U}^+\nabla_\mu\bar{U}\nabla^\nu\bar{U}^+\nabla_\nu\bar{U}\rangle$$
$$-iN\langle F^R_{\mu\nu}\nabla^\mu\bar{U}\nabla^\nu\bar{U}^+\rangle - iN\langle F^L_{\mu\nu}\nabla^\mu\bar{U}^+\nabla^\nu\bar{U}\rangle$$
$$-N\langle F^R_{\mu\nu}\bar{U}F^{\mu\nu L}\bar{U}^+\rangle - \frac{N}{2}\langle F^R_{\mu\nu}F^{\mu\nu R} + F^L_{\mu\nu}F^{\mu\nu L}\rangle \tag{7.21}$$

whereas the explicit expression for $\mathrm{Sp}\hat{\sigma}^2$ becomes

$$\mathrm{Sp}\hat{\sigma}^2 = \frac{1}{8}\langle\nabla^\mu\bar{U}^+\nabla_\mu\bar{U}\rangle^2 + \frac{1}{4}\langle\nabla^\mu\bar{U}^+\nabla^\nu\bar{U}\rangle\langle\nabla_\mu\bar{U}^+\nabla_\nu\bar{U}\rangle$$
$$+\frac{N}{8}\langle\nabla^\mu\bar{U}^+\nabla_\mu\bar{U}\nabla^\nu\bar{U}^+\nabla_\nu\bar{U}\rangle + \frac{1}{4}\langle\nabla^\mu\bar{U}^+\nabla_\mu\bar{U}\rangle\langle\chi^+\bar{U}+\bar{U}^+\chi\rangle$$
$$+\frac{N}{4}\langle\nabla^\mu\bar{U}^+\nabla_\mu\bar{U}(\chi^+\bar{U}+\bar{U}^+\chi)\rangle + \frac{N^2+2}{8N^2}\langle\chi^+\bar{U}+\bar{U}^+\chi\rangle^2$$
$$+\frac{N^2-4}{8N}\langle\chi^+\bar{U}\chi^+\bar{U}+\bar{U}^+\chi\bar{U}^+\chi + 2\chi^+\chi\rangle \tag{7.22}$$

The pole at d = 4 in ln det D thus has the form of the general gauge invariant Lagrangian L_2 given in section 6 - with the exception of the term $\langle\nabla^\mu U^+\nabla^\nu U\nabla_\mu U^+\nabla_\nu U\rangle$ ocurring in $\mathrm{Sp}\hat{\Gamma}_{\mu\nu}\hat{\Gamma}^{\mu\nu}$. Indeed, for arbitrary N the general gauge invariant Lagrangian of order p^4 contains an additional term. For N = 3 we however have the following identity

$$\langle ABAB\rangle = -2\langle A^2B^2\rangle + \frac{1}{2}\langle A^2\rangle\langle B^2\rangle + \langle AB\rangle^2 \tag{7.23}$$

valid for any pair of traceless, hermitean 3x3 matrices A, B. Applying this identity with

$$A = i\nabla_\mu U^+U \quad , \quad B = iU^+\nabla_\nu U$$

we obtain

$$\langle\nabla^\mu U^+\nabla^\nu U\nabla_\mu U^+\nabla_\nu U\rangle = -2\langle\nabla^\mu U^+\nabla_\mu U\nabla^\nu U^+\nabla_\nu U\rangle$$
$$+\frac{1}{2}\langle\nabla^\mu U^+\nabla_\mu U\rangle^2 + \langle\nabla^\mu U^+\nabla^\nu U\rangle\langle\nabla_\mu U^+\nabla_\nu U\rangle \tag{7.24}$$

For N = 3 the pole at d = 4 in ln det D may therefore be absorbed by the following renormalization of the low energy coupling constants

$$
\begin{aligned}
& L_i = L_i^r + \Gamma_i \lambda \qquad i = 1,\dots,10 \\
& H_i = H_i^r + \Delta_i \lambda \qquad i = 1,2 \\
& \lambda = (4\pi)^{-2} \mu^{d-4} \left\{ \frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1 \right) \right\} \qquad (7.25) \\
& \Gamma_1 = \frac{3}{32}, \; \Gamma_2 = \frac{3}{16}, \; \Gamma_3 = 0, \; \Gamma_4 = \frac{1}{8}, \; \Gamma_5 = \frac{3}{8} \\
& \Gamma_6 = \frac{11}{144}, \; \Gamma_7 = 0, \; \Gamma_8 = \frac{5}{48}, \; \Gamma_9 = \frac{1}{4}, \; \Gamma_{10} = -\frac{1}{4} \\
& \Delta_1 = -\frac{1}{8}, \; \Delta_2 = \frac{5}{24}
\end{aligned}
$$

Expressed in terms of the renormalized coupling constants L_i^r, H_i^r the sum $Z_2 + Z_{\text{one loop}}$ remains finite at d = 4.

8. Tadpoles and unitarity corrections

To work out the explicit contribution of the one loop graphs to a given Green's function we need to calculate the determinant of the differential operator D in a series expansion in powers of the external fields. If these fields are switched off, D reduces to

$$D_0^{ab} = \delta^{ab} \Box + \frac{1}{2} B_0 \, \mathrm{tr}\left(\{ \lambda^a, \lambda^b \} m \right) \qquad (8.1)$$

If m_u and m_d are different, then this operator is not diagonal in the cartesian basis spanned by $\lambda^1, \dots, \lambda^8$. It is convenient to instead use the physical basis $\lambda_{\pi^+}, \dots, \lambda_\eta$ defined in (5.16), in which we have

$$D_{0PQ} = \delta_{PQ} \left\{ \Box + \overset{\circ}{M}{}_P^2 \right\} \qquad (8.2)$$

In this basis the full operator D is given by

$$\begin{aligned} D &= D_0 + \delta \\ \delta &= \{\hat{\Gamma}^\mu, \partial_\mu\} + \hat{\Gamma}^\mu \hat{\Gamma}_\mu + \bar{\sigma} \end{aligned} \tag{8.3}$$

with

$$\begin{aligned} \hat{\Gamma}^\mu_{PQ} &= -\tfrac{1}{2}\, tr\left([\lambda_P, \lambda_Q^+]\Gamma_\mu\right) \\ \bar{\sigma}_{PQ} &= \sigma^\Delta_{PQ} + \sigma^\chi_{PQ} \\ \sigma^\Delta_{PQ} &= \tfrac{1}{2}\, tr\left([\lambda_P, \Delta_\mu][\lambda_Q^+, \Delta^\mu]\right) \\ \sigma^\chi_{PQ} &= \tfrac{1}{8}\, tr\left(\{\lambda_P, \lambda_Q^+\}(u\chi^+ u + u^+ \chi u^+)\right) - \delta_{PQ} \overset{\circ}{M}{}^2_P \end{aligned} \tag{8.4}$$

The 3x3 matrices Γ_μ and Δ_μ are defined in (7.4). Since δ vanishes if the external fields are switched off, we may expand ln det D in powers of δ

$$Z_{one\ loop} = \frac{i}{2} \ln \det D_0 + \frac{i}{2} Tr(D_0^{-1}\delta) - \frac{i}{4} Tr(D_0^{-1}\delta D_0^{-1}\delta) + \dots \tag{8.5}$$

where Tr denotes the trace in flavour and coordinate space. The term $Tr(D_0^{-1}\delta)$ is the set of all tadpole graphs (loop interrupted only at one point). The next term collects all graphs with two vertices in the loop etc. If we count the external fields $a_\mu(x)$ and $p(x)$ as quantities of order ϕ, $v_\mu(x)$ and $s(x) - M$ as order ϕ^2, then Δ_μ is $O(\phi)$ whereas Γ_μ and $\bar{\sigma}$ are $O(\phi^2)$. The operator δ is therefore of order ϕ^2. If we stop the expansion (8.5) at the term $Tr(D_0^{-1}\delta D_0^{-1}\delta)$ then we obtain all one loop graphs which contribute to the generating functional up to and including $O(\phi^4)$. This accuracy suffices to calculate all two-point functions, the vertex functions containing at most one vector or scalar current, as well as the four-point functions of A^a_μ and P^a.

More explicitly, the representation (8.5) is of the form

$$
\begin{aligned}
Z_{\text{one loop}} = \frac{i}{2} \sum_P \Delta_P(0) \int dx\, \overline{\sigma}_{PP}(x) \\
+ \sum_{P,Q} \int dx\,dy \Big\{ M_{\mu\nu}(x-y)\, \hat{\Gamma}^{\mu}_{PQ}(x)\, \hat{\Gamma}^{\nu}_{QP}(y) \\
+ K_\mu(x-y)\, \hat{\Gamma}^{\mu}_{PQ}(x)\, \overline{\sigma}_{QP}(y) \\
+ \frac{1}{4} J(x-y)\, \overline{\sigma}_{PQ}(x)\, \overline{\sigma}_{QP}(y) \Big\} + O(\phi^6)
\end{aligned}
$$

$$
\begin{aligned}
M_{\mu\nu}(z) &= \frac{i}{4} \Big\{ \partial_\mu \Delta_P \partial_\nu \Delta_Q + \partial_\nu \Delta_P \partial_\mu \Delta_Q - \partial_{\mu\nu} \Delta_P \Delta_Q - \Delta_P \partial_{\mu\nu} \Delta_Q \\
&\quad + g_{\mu\nu}\, \delta(z) \left(\Delta_P(0) + \Delta_Q(0) \right) \Big\} \\
K_\mu(z) &= \frac{i}{2} \left\{ \partial_\mu \Delta_P \Delta_Q - \Delta_P \partial_\mu \Delta_Q \right\} \\
J(z) &= -i\, \Delta_P \Delta_Q
\end{aligned}
\tag{8.6}
$$

where $\Delta_P(z) = \Delta_c(z, M_P^2)$ is the Feynman propagator for a scalar field of mass M_P in d dimensions. The kernels $M^{\mu\nu}$ and J have poles at d = 4:

$$
\begin{aligned}
M_{\mu\nu}(z) &= (\partial_{\mu\nu} - g_{\mu\nu}\Box) M^r(z) - g_{\mu\nu} L(z) - \frac{1}{6}\lambda (\partial_{\mu\nu} - g_{\mu\nu}\Box)\delta(z) \\
K_\mu(z) &= -\partial_\mu K(z) \\
J(z) &= J^r(z) - 2\lambda\delta(z)
\end{aligned}
\tag{8.7}
$$

The poles are contained in the quantity λ defined in (7.25). The Fourier transform of the scalars J^r, K, L, M^r, normalized by

$$
J^r(s) = \int dz\, e^{ipz} J^r(z) \quad ; \quad s = p^2
$$

may be expressed in terms of the function $\bar{J}(s)$

$$\bar{J}(s) = (32\pi^2)^{-1}\left\{2 + \frac{\Delta}{s}\ln\frac{M_Q^2}{M_P^2} - \frac{\Sigma}{\Delta}\ln\frac{M_Q^2}{M_P^2} - \frac{\nu}{s}\ln\frac{(s+\nu)^2-\Delta^2}{(s-\nu)^2-\Delta^2}\right\}$$

$$\nu^2 = s^2 + M_P^4 + M_Q^4 - 2s(M_P^2 + M_Q^2) - 2M_P^2 M_Q^2$$

$$\Sigma = M_P^2 + M_Q^2 \quad , \quad \Delta = M_P^2 - M_Q^2 \tag{8.8}$$

(See the appendix for a discussion of the relations (8.8)-(8.10).) One finds

$$J^r = \bar{J} - 2k$$

$$K = \frac{\Delta}{2s}\bar{J}$$

$$L = \frac{\Delta^2}{4s}\bar{J}$$

$$M^r = \frac{1}{12s}\{s - 2\Sigma\}\bar{J} + \frac{\Delta^2}{3s^2}\bar{\bar{J}} - \frac{1}{6}k + \frac{1}{288\pi^2} \tag{8.9}$$

with

$$k = \frac{1}{32\pi^2}\,\frac{M_P^2\ln\frac{M_P^2}{\mu^2} - M_Q^2\ln\frac{M_Q^2}{\mu^2}}{M_P^2 - M_Q^2}$$

$$\bar{\bar{J}}(s) = \bar{J}(s) - s\bar{J}'(0) \tag{8.10}$$

Note that all of these functions depend on s, M_P^2 and M_Q^2 and, in the case of J^r, M^r, on the renormalization scale μ occurring in the constant k.

The generating functional Z, obtained by adding $\int dx(\mathcal{L}_1 + \mathcal{L}_2) + Z_A$ to $Z_{\text{one loop}}$ then takes the form

$$Z = Z_t + Z_u + Z_A + O(\phi^6) \tag{8.11}$$

Z_t denotes the sum of tree graph and tadpole contributions

$$\begin{aligned} Z_t &= \sum_P \int dx \frac{F_0^2}{6}\left\{1 - \frac{3}{16\pi^2}\frac{M_P^2}{F_0^2}\ln\frac{M_P^2}{\mu^2}\right\}\sigma_{PP}^{\Delta} \\ &+ \sum_P \int dx \frac{3F_0^2}{16}\left\{1 - \frac{1}{6\pi^2}\frac{M_P^2}{F_0^2}\ln\frac{M_P^2}{\mu^2}\right\}\sigma_{PP}^{\chi} \\ &+ \int dx\, \mathcal{L}_2^r \end{aligned} \tag{8.12}$$

where $\mathcal{L}_2^r$ is given by (6.16), except that the constants L_i, H_i are replaced by the renormalized quantities L_i^r, H_i^r. The "unitarity correction" Z_u contains one loop graphs with two vertices,

$$\begin{aligned} Z_u = \sum_{P,Q} \int dx dy \Big[&\{\partial_{\mu\nu} - g_{\mu\nu}\Box\} M^r(x-y) - g_{\mu\nu} L(x-y)\} \hat{\Gamma}_{PQ}^{\mu}(x)\hat{\Gamma}_{QP}^{\nu}(y) \\ &- \partial_\mu K(x-y)\hat{\Gamma}_{PQ}^{\mu}(x)\bar{\sigma}_{QP}(y) + \frac{1}{4} J^r(x-y)\bar{\sigma}_{PQ}(x)\bar{\sigma}_{QP}(y)\Big] \end{aligned} \tag{8.13}$$

Finally the anomaly Z_A is given in eq. (6.11).

To arrive at this result, note that the contribution from the difference $M_{\mu\nu} - M_{\mu\nu}^r$ is proportional to $\mathrm{Sp}(\partial_\mu\hat{\Gamma}_\nu - \partial_\nu\hat{\Gamma}_\mu)^2$. Since $\hat{\Gamma}_\mu$ is of order ϕ^2, this quantity coincides with $\mathrm{Sp}\hat{\Gamma}_{\mu\nu}^2$ at the order we are considering here. Furthermore, extracting the pole in $\Delta_P(0)$ (see the appendix)

$$-i\Delta_P(0) = 2M_P^2\lambda + \frac{1}{16\pi^2}M_P^2\ln\frac{M_P^2}{\mu^2} \tag{8.14}$$

and using the identities

$$\sum_{P,Q} \bar{\sigma}_{PQ}\bar{\sigma}_{QP} = Sp\hat{\sigma}^2 - 2\sum_P M_P^2 \bar{\sigma}_{PP} + 2\sum_P M_P^4$$

(8.15)

$$tr\{\nabla_\mu \bar{U}^+ \nabla^\mu \bar{U} + \chi^+ \bar{U} + \chi \bar{U}^+\} = \sum_P \left\{\frac{2}{3}\sigma_{PP}^{\Delta} + \frac{3}{4}\sigma_{PP}^{\chi}\right\} + \frac{3}{4}\sum_P M_P^2$$

one shows that the tree graph contibution $\int dx(L_1 + L_2)$ merges with the tadpole term proportional to $\Delta_P(0)$ and with the contribution from the difference $J - J^r$ into the expression for Z_t given in (8.12).

9. Vacuum expectation values

With the explicit expression for the generating functional given in the preceding section it is a simple matter to determine the vacuum expectation values of $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$ up to and including contributions of order M: it suffices to set $s(x) = M + \delta s(x)$ [$v_\mu(x) = a_\mu(x) = p(x) = 0$; in the absence of these external fields $\bar{U} = 1$, independently of δs] and to extract the coefficient of the term linear in $\delta s(x)$. The result reads (Langacker and Pagels 1973; Novikov et al. 1981)

$$\langle 0|\bar{u}u|0\rangle = -F_0^2 B_0\{1 - 2\mu_{\pi^+} - 2\mu_{K^+} - \mu_{\pi^0}(\cos\varepsilon + \tfrac{1}{\sqrt{3}}\sin\varepsilon)^2$$
$$- \mu_\eta(-\sin\varepsilon + \tfrac{1}{\sqrt{3}}\cos\varepsilon)^2 + m_u K_1 + K_2\}$$
$$\langle 0|\bar{d}d|0\rangle = -F_0^2 B_0\{1 - 2\mu_{\pi^+} - 2\mu_{K^0} - \mu_{\pi^0}(\cos\varepsilon - \tfrac{1}{\sqrt{3}}\sin\varepsilon)^2$$
$$- \mu_\eta(\sin\varepsilon + \tfrac{1}{\sqrt{3}}\cos\varepsilon)^2 + m_d K_1 + K_2\}$$
$$\langle 0|\bar{s}s|0\rangle = -F_0^2 B_0\{1 - 2\mu_{K^+} - 2\mu_{K^0} - \tfrac{4}{3}(\mu_{\pi^0}\sin^2\varepsilon + \mu_\eta\cos^2\varepsilon)$$
$$+ m_s K_1 + K_2\} \qquad (9.1)$$

with

$$K_1 = 8B_0F_0^{-2}(2L_8^r + H_2^r) \quad ; K_2 = (m_u + m_d + m_s)32B_0F_0^{-2}L_6^r$$
$$\mu_P = (32\pi^2)^{-1}M_P^2F_0^{-2}\ln\frac{M_P^2}{\mu^2} \qquad (9.2)$$

Note that the scale dependence of the chiral logarithms μ_P cancels the scale dependence of the coupling constants L_6^r, L_8^r, H_2^r - the expectation values are independent of the chiral scale μ, as they should be.

The ratio $\langle\bar{u}u\rangle/\langle\bar{s}s\rangle$ measures the vacuum asymmetry induced by the quark mass term. Neglecting the difference between m_u and m_d this ratio is given by

$$\frac{\langle 0|\bar{s}s|0\rangle}{\langle 0|\bar{u}u|0\rangle} = 1 + 3\mu_\pi - 2\mu_K - \mu_\eta + (m_s - \hat{m})K_1 \qquad (9.3)$$

Since the constant K_1 contains the contact term H_2, its value depends on the conventions used to specify the quantity $\langle 0|\bar{q}q|0\rangle$. There is no ambiguity in the vacuum expectation values to leading order, but the first order perturbation theory formula

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}q|0\rangle_0 - i\int dx \langle 0|T\,\bar{q}Mq\,\bar{q}q|0\rangle + O(M^2) \qquad (9.4)$$

requires subtractions to converge - $\langle 0|\bar{q}q|0\rangle$ depends on the manner in which one subtracts. The same ambiguity occurs in the ratio $\langle\bar{d}d\rangle/\langle\bar{u}u\rangle$. Eliminating the constant H_2 one obtains the following sum rule connecting the isospin asymmetry $\langle\bar{d}d\rangle/\langle\bar{u}u\rangle$ to the SU(3) asymmetry $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$

$$\frac{\langle 0|\bar{d}d|0\rangle}{\langle 0|\bar{u}u|0\rangle} = 1 - \frac{m_d - m_u}{m_s - \hat{m}}\left\{1 - \frac{\langle 0|\bar{s}s|0\rangle}{\langle 0|\bar{u}u|0\rangle} + \frac{1}{16\pi^2 F_0^2}\left(M_K^2 - M_\pi^2 - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right)\right\} \qquad (9.5)$$

where we have dropped terms of order $(m_u - m_d)^2$. (The sum rule holds in any subtraction convention consistent with chiral symmetry.) If the expectation value of $\bar{s}s$ is smaller than the expectation value of $\bar{u}u$ then this relation implies $|\langle\bar{d}d\rangle| < |\langle\bar{u}u\rangle|$. Using the numerical values $F_0 \simeq F_\pi = 93.3$ MeV, $(m_s - \hat{m})/(m_d - m_u) = 43.5$ we predict that $|\langle\bar{d}d\rangle|$ is smaller than $|\langle\bar{u}u\rangle|$ by (a) 0.3%, (b) 0.6%, (c) 1% if $|\langle\bar{s}s\rangle|$ is smaller than $|\langle\bar{u}u\rangle|$ by (a) 0%, (b) 15%, (c) 30%, respectively. Values for the asymmetries in the vacuum expectation values, obtained on the basis of QCD sum rules are given in (Shifman, Vainshtein and Zakharov 1979; Ioffe 1981; Ioffe and Belyaev 1982; Mallik 1982; Pascual and Tarrach 1982; Reinders, Rubinstein and Yazaki 1983; Narison, Paver and Treleani 1983; Bagan, Bramon, Narison and Paver 1983).

10. Masses and decay constants

To extract the two-point functions of the axial current from the low energy representation (8.11,12,13) for the generating functional we expand it to second order in the external fields a_μ, switching all other external sources off. The explicit expression for Z contains the external field a_μ both explicitly, and implicitly, through the meson field ϕ which is determined by a_μ through the equations of motion (5.14). The calculation of the Green's functions simplifies considerably if we make use of the following observation. The equations of motion state that the meson field realizes an extremum of the classical action $\int dx L_1$. This implies that if we modify the field ϕ by $\delta\phi$ we change the value of $\int dx L_1$ only by an amount of order $(\delta\phi)^2$. In particular, if we determine the field ϕ by

$$\frac{\delta Z}{\delta \varphi} = 0 \tag{10.1}$$

(rather than by the extremum of the lowest order contribution $\int dx L_1$ to Z) we commit an error in ϕ of relative order p^2 and hence generate an error in the value of $\int dx L_1$ of relative order p^4. This error is beyond the accuracy of our low energy representation - we may therefore use (10.1) instead of the original equations of motion. This step automatically eliminates the double poles generated by mass renormalization.

In the case of the two-point function of the axial current only tree graphs and tadpoles described by Z_t contribute (since a_μ counts as $O(\phi)$ we need Z to order ϕ^2; Z_u is of order ϕ^4, Z_A is of order ϕ^3). The positions (residues) of the poles in the two-point function determine the masses (decay constants). For the evaluation of these pole terms the contributions proportional to $(\partial_\mu a_\nu - \partial_\nu a_\mu)^2$ generated by the constants L_{10} and H_1 may be disregarded. The terms linear or quadratic in the meson field are of the form (the components ϕ^a refer to the octet basis)

$$Z = \frac{1}{2}\int dx \sum_{a,b} \left\{ A^{ab} (\partial_\mu \varphi^a - a_\mu^a)(\partial^\mu \varphi^b - a^{\mu b}) - B^{ab} \varphi^a \varphi^b \right\} + \dots \tag{10.2}$$

where the matrices A and B may be worked out from (8.12). The extremum of Z is at

$$A \Box \varphi + B \varphi = A \partial_\mu a^\mu \tag{10.3}$$

To solve this equation of motion we diagonalize B relative to A, i.e. introduce a matrix F such that

$$A = F^{+} \mathbb{1} F \quad , \quad B = F^{+} M^{2} F \tag{10.4}$$

where M^2 is diagonal. The solution then reads

$$\varphi = F^{-1} (\Box + M^{2})^{-1} F \partial_{\mu} a^{\mu} \tag{10.5}$$

and the value of the generating functional at the extremum becomes

$$Z = \frac{1}{2} \int dx \sum_{P} \alpha^{P}(x)^{*} \Delta(x-y; M_{P}^{2}) \alpha^{P}(y) + \dots$$

$$\alpha^{P}(x) = \sum_{a} F^{Pa} \partial_{\mu} a^{\mu a}(x)$$

The pole contribution to the two-point function is therefore given by

$$\langle 0 | T A_{\mu}^{a}(x) A_{\nu}^{b}(y) | 0 \rangle = i \sum_{P} \partial_{\mu\nu} \Delta(x-y; M_{P}^{2}) F^{Pa*} F^{Pb}$$

This shows that the eigenvalues M_P of the diagonal matrix M defined in (10.4) are the physical masses, whereas the one particle matrix elements of the axial currents are given by the matrix F:

$$\langle 0 | A_{\mu}^{a} | P, p \rangle = i p_{\mu} F^{Pa} \tag{10.6}$$

With the explicit form of the matrices A and B which follows from the representation (8.12) of the tree graph and tadpole contributions, the masses and decay constants are easily worked out. Neglecting the mass difference $m_u - m_d$ we obtain

$$M_\pi^2 = 2\hat{m} B_0 \left\{ 1 + \mu_\pi - \tfrac{1}{3}\mu_\eta + 2\hat{m} K_3 + K_4 \right\}$$

$$M_K^2 = (\hat{m} + m_s) B_0 \left\{ 1 + \tfrac{2}{3}\mu_\eta + (\hat{m} + m_s) K_3 + K_4 \right\}$$

$$M_\eta^2 = \tfrac{2}{3}(\hat{m} + 2m_s) B_0 \left\{ 1 + 2\mu_K - \tfrac{4}{3}\mu_\eta + \tfrac{2}{3}(\hat{m} + 2m_s) K_3 + K_4 \right\}$$
$$+ 2\hat{m} B_0 \left\{ -\mu_\pi + \tfrac{2}{3}\mu_K + \tfrac{1}{3}\mu_\eta \right\} + K_5 \tag{10.7}$$

$$F_\pi = F_0 \left\{ 1 - 2\mu_\pi - \mu_K + 2\hat{m} K_6 + K_7 \right\}$$

$$F_K = F_0 \left\{ 1 - \tfrac{3}{4}\mu_\pi - \tfrac{3}{2}\mu_K - \tfrac{3}{4}\mu_\eta + (\hat{m} + m_s) K_6 + K_7 \right\}$$

$$F_\eta = F_0 \left\{ 1 - 3\mu_K + \tfrac{2}{3}(\hat{m} + 2m_s) K_6 + K_7 \right\}$$

where the constants K_i denote the following combinations of the low energy coupling constants

$$K_3 = \frac{8B_0}{F_0^2}(2L_8^r - L_5^r)$$

$$K_4 = (m_u + m_d + m_s)\frac{16B_0}{F_0^2}(2L_6^r - L_4^r)$$

$$K_5 = (m_s - \hat{m})^2 \frac{128}{9}\frac{B_0^2}{F_0^2}(3L_7 + L_8^r) \tag{10.8}$$

$$K_6 = \frac{4B_0}{F_0^2} L_5^r$$

$$K_7 = (m_u + m_d + m_s)\frac{8B_0}{F_0^2} L_4^r$$

The quantity μ_p is defined in (9.2). As a check one verifies that the masses and decay constants are independent of the renormalization scale μ. (The nonanalytic pieces in these expansions coincide with those found by Langacker and Pagels (1973), if one replaces $\ln M_p^2/\mu^2$ by $\ln M^2/\mu^2$ where M denotes a common meson mass.)

At leading order in the quark mass expansion the Gell-Mann-Okubo formula holds exactly; at first nonleading order the formula receives a correction. Defining Δ_{GMO} by

$$\Delta_{GMO} \equiv (4M_K^2 - M_\pi^2 - 3M_\eta^2)/(M_\eta^2 - M_\pi^2) \tag{10.9}$$

we obtain

$$\begin{aligned}\Delta_{GMO} = &-2(4M_K^2\mu_K - M_\pi^2\mu_\pi - 3M_\eta^2\mu_\eta)/(M_\eta^2 - M_\pi^2) \\ &- \frac{6}{F_0^2}(M_\eta^2 - M_\pi^2)\{12L_7 + 6L_8^r - L_5^r\}\end{aligned} \tag{10.10}$$

(Experimentally, correcting the masses for electromagnetic effects, one finds $\Delta_{GMO} = 0.21$.)

Likewise, the ratio $M_K^2 : M_\pi^2$ is equal to the quark mass ratio $(m_s + \hat{m}) : 2\,\hat{m}$ only at leading order. The mass formulae (10.7) imply the following correction

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\{1 + \Delta_M\} \tag{10.11}$$

$$\Delta_M = -\mu_\pi + \mu_\eta + \frac{8}{F_0^2}(M_K^2 - M_\pi^2)(2L_8^r - L_5^r)$$

The ratio $F_K : F_\pi$ is given by

$$\begin{aligned}\frac{F_K}{F_\pi} &= 1 + \Delta_F \\ \Delta_F &= \frac{5}{4}\mu_\pi - \frac{1}{2}\mu_K - \frac{3}{4}\mu_\eta + \frac{4}{F_0^2}(M_K^2 - M_\pi^2)L_5^r\end{aligned} \tag{10.12}$$

Eliminating the constants K_6 and K_7 in (10.7) we obtain the following sum rule expressing F_η in terms of F_π and F_K:

$$F_\eta = F_\pi \left(\frac{F_K}{F_\pi}\right)^{\frac{4}{3}} \left\{ 1 + \frac{1}{96\pi^2 F_0^2} \left(3M_\eta^2 \ln \frac{M_\eta^2}{M_K^2} - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2}\right) + O(m^2) \right\} \quad (10.13)$$

With the experimental value (Leutwyler and Roos, 1984)

$$\frac{F_K}{F_\pi} = 1.22 \pm 0.01 \quad (10.14)$$

the sum rule (10.13) predicts

$$\frac{F_\eta}{F_\pi} = 1.3 \pm 0.05 \quad (10.15)$$

The quark mass difference $m_u - m_d$ generates the following first order isospin breaking effects in the Kaon system

$$(M_{K^0}^2 - M_{K^+}^2)_{QCD} = (m_d - m_u) B_0 \left\{ 1 + \frac{2}{3}\mu_\eta + M_K^2 \frac{\mu_\eta - \mu_\pi}{M_K^2 - M_\pi^2} + 2(m_s + \hat{m}) K_3 + K_4 \right\} + O(m^3)$$

$$\left(\frac{F_{K^0}}{F_{K^+}}\right)_{QCD} = 1 - (m_d - m_u) B_0 \left\{ \frac{1}{64\pi^2 F_0^2} \left(\ln \frac{M_K^2}{\mu^2} + 1\right) + \frac{\mu_\eta - \mu_\pi}{M_\eta^2 - M_\pi^2} \right\} + (m_d - m_u) K_6 + O(m^2) \quad (10.16)$$

At leading order in the quark mass expansion the ratio $(M_{K^0}^2 - M_{K^+}^2) : (M_K^2 - M_\pi^2)$ is given by $(m_d - m_u) : (m_s - \hat{m})$. Comparing (10.16) with (10.7) we find the following correction to this relation

$$\frac{(M_{K^0}^2 - M_{K^+}^2)_{QCD}}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} \{1 + \Delta_M\} \tag{10.17}$$

where Δ_M is defined in (10.11): the correction factor is the same as in the relation for $M_K^2 : M_\pi^2$. We therefore obtain a sum rule which does not involve unkown low energy constants

$$\frac{(M_{K^0}^2 - M_{K^+}^2)_{QCD}}{M_K^2 - M_\pi^2} \cdot \frac{M_\pi^2}{M_K^2} = \frac{m_d - m_u}{m_s - \hat{m}} \cdot \frac{2\hat{m}}{m_s + \hat{m}} \cdot \{1 + O(m^2)\} \tag{10.18}$$

Since the ratio $(m_d - m_u) : (m_s - \hat{m})$, which compares isospin breaking with SU(3) breaking, is known very accurately (see I) this sum rule may be used to obtain a value for the ratio $m_s : \hat{m}$ which is not known with the same precision. To extract this information we first correct the mass difference between K^0 and K^+ for electromagnetic effects with Dashen's theorem (Dashen 1969)

$$(M_{K^0}^2 - M_{K^+}^2)_{QCD} = M_{K^0}^2 - M_{K^+}^2 - M_{\pi^0}^2 + M_{\pi^+}^2 + O(e^2 m) \tag{10.19}$$

Next, in order only to use independent information, we determine the ratio $R = (m_s - \hat{m}) : (m_d - m_u)$ from isospin breaking in the baryon mass spectrum and from ρ-ω-mixing. This is easily done on the basis of table 4 given in ref. I. Eliminating the contribution from $K^0 - K^+$ in this table one finds $R = 43.7 \pm 2.7$. Inserting this number in (10.18) and using (10.19) we obtain

$$\frac{m_s}{\hat{m}} = 25.7 \pm 2.6 \tag{10.20}$$

where the error includes the uncertainties due to effects of order M^2 in (10.18) as well as the uncertainties due to the corrections of order e^2M to the low energy theorem (10.19). We thus confirm the value $m_s : \hat{m} = 25.0 \pm 2.5$ obtained in I from the mass ratios $M_\pi^2 : M_K^2 : M_\eta^2$, i.e. from independent experimental information.

For later use we note that the two ratios $(M_{K^0}^2 - M_{K^+}^2)_{QCD} : (M_K^2 - M_\pi^2)$ and $(m_d - m_u) : (m_s - \hat{m})$ turn out to be the same within rather small errors. This implies that the quantity Δ_M defined in (10.17) is small:

$$\Delta_M = \pm 0.09 \tag{10.21}$$

In order to evaluate the isospin asymmetry in the constants F_{K^0}, F_{K^+} which follows from (10.16) we express the constant K_6 in terms of $F_K : F_\pi$ with the result

$$\left(\frac{F_{K^0}}{F_{K^+}}\right)_{QCD} = 1 + \frac{m_d - m_u}{m_s - \hat{m}} \left\{ \frac{F_K}{F_\pi} - 1 - \frac{1}{64\pi^2 F_0^2} \left(M_K^2 - M_\pi^2 - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2} \right) \right\} + O(m^2) \tag{10.22}$$

With the experimental value for $F_K : F_\pi$ given above this sume rule predicts that the isospin breaking is very small

$$\left(\frac{F_{K^0}}{F_{K^+}}\right)_{QCD} = 1.004 \tag{10.23}$$

Furthermore, $m_u \neq m_d$ induces $\pi^0\eta$ mixing. To first order in $m_u - m_d$ the contributions to the generating functional which are linear or quadratic in the fields ϕ^3 and ϕ^8 may be written in the form ($v_\mu = p = 0$, $s = M$):

$$\begin{aligned} Z = \frac{1}{2} \int dx \Big\{ & F_\pi^2 (\nabla_\mu \varphi^3 + \varepsilon_1 \nabla_\mu \varphi^8)^2 + F_\eta^2 (\nabla_\mu \varphi^8 - \varepsilon_2 \nabla_\mu \varphi^3)^2 \\ & - F_\pi^2 M_\pi^2 (\varphi^3 + \varepsilon_1 \varphi^8)^2 - F_\eta^2 M_\eta^2 (\varphi^8 - \varepsilon_2 \varphi^3)^2 \Big\} + \dots \end{aligned}$$

$$\nabla_\mu \varphi^a = \partial_\mu \varphi^a - a_\mu^a \tag{10.24}$$

where M_π, M_η, F_π, F_η are the isospin symmetric masses and decay constants given in (10.7). The two mixing angles ε_1, ε_2 are given by

$$\varepsilon_1 = \varepsilon_2 + \frac{\sqrt{3}}{2}\,\frac{m_d - m_u}{m_s - \hat{m}}\,(M_K^2 - M_\pi^2)\,C_1$$

$$\varepsilon_2 = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s - \hat{m}}\left\{1 - 3\mu_\pi + 2\mu_K + \mu_\eta + M_\pi^2 C_1 - \frac{32}{F_0^2}(M_K^2 - M_\pi^2)(3L_7 + L_8^r)\right\}$$

$$C_1 = \frac{1}{16\pi^2 F_0^2}\left(1 - \frac{M_\pi^2}{M_K^2 - M_\pi^2}\,\ln\frac{M_K^2}{M_\pi^2}\right) \tag{10.25}$$

They determine the off-diagonal matrix elements

$$\begin{aligned} \langle 0|A_\mu^8|\pi^0\rangle &= i\,p_\mu\,\varepsilon_1 F_\pi \\ \langle 0|A_\mu^3|\eta\rangle &= -i\,p_\mu\,\varepsilon_2 F_\eta \end{aligned} \tag{10.26}$$

Note that ε_1 and ε_2 are different. The notion of a $\pi^0\eta$ mixing angle is a well-defined concept only at leading order in the quark mass expansion; at the order we are considering here there is no angle ε for which the two orthogonal combinations $\cos\varepsilon\, A_\mu^3 + \sin\varepsilon\, A_\mu^8$ and $-\sin\varepsilon\, A_\mu^3 + \cos\varepsilon\, A_\mu^8$ would have nonvanishing matrix elements only with the states $|\pi^0\rangle$ and $|\eta\rangle$ respectively. To evaluate ε_2 we express the constant $3\,L_7 + L_8^r$ in terms of $\Delta_{GMO} + \Delta_M$. Using (10.17) we obtain

$$\begin{aligned} \varepsilon_2 &= \frac{\sqrt{3}}{4}\,\frac{(M_{K^0}^2 - M_{K^+}^2)_{QCD}}{M_K^2 - M_\pi^2}\left\{1 + \Delta_{GMO} + C_1 M_\pi^2 + C_2\right\} \\ C_2 &= \frac{1}{16\pi^2 F_0^2}\,\frac{M_\eta^2}{M_\eta^2 - M_\pi^2}\left\{3M_\eta^2 \ln\frac{M_K^2}{M_\eta^2} + M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right\} \end{aligned} \tag{10.27}$$

With the value $(M_{K^0} - M_{K^+})_{QCD} = 5.28$ MeV given in I this becomes

$$\begin{aligned} \varepsilon_1 &= 1.37\cdot 10^{-2} = 0.78^\circ \\ \varepsilon_2 &= 1.11\cdot 10^{-2} = 0.63^\circ \end{aligned} \tag{10.28}$$

to be compared with the lowest order mixing angle

$$\varepsilon = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s - \hat{m}} = 1.00\cdot 10^{-2} = 0.57^\circ \tag{10.29}$$

The differences $M^2_{\pi^+} - M^2_{\pi^0}$ and $F_{\pi^+} - F_{\pi^0}$ (where F_{π^0} measures the π^0 matrix element of A^3_μ) are of second order in $m_u - m_d$ and are therefore tiny. We did not work out the isospin breaking in the decay constants. The mass difference $M_{\pi^+} - M_{\pi^0}$ is discussed at the end of the next section.

11. Comparison with SU(2)xSU(2)

The low energy expansion simplifies considerably if one limits the external momenta to values small compared to M_K, M_η and treats m_u, m_d as small in comparison to m_s

$$p^2 \ll M_K^2 \;\; ; \;\; m_u, m_d \ll m_s \tag{11.1}$$

In this region the degrees of freedom of the K and η mesons freeze.

If we only consider Green's functions involving u or d quarks and, furthermore, ignore the isoscalar currents $\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu u$, $\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d$, the generating functional at order p^4 reduces in the limit (11.1) to the corresponding low energy expansion for SU(2)xSU(2) which we analysed in detail in II. In particular the seven low energy constants $\ell_1, \dots, \ell_7$ and the three high energy constants h_1, h_2 and h_3 which specify the general effective Lagrangian of SU(2)xSU(2) at order p^4 can be expressed in terms of the parameters $L_1, \dots, L_{10}$, H_1 and H_2. With these relations at hand, the phenomenological information on the values of the low energy constants $\ell_1, \dots, \ell_7$ obtained in II may then be translated into information concerning the SU(3)xSU(3) coupling constants. (Since the SU(3)xSU(3) Lagrangian contains three additional low energy coupling constants we however need additional information, either from data on K_{ℓ_4} decay (or low energy $K\pi$ scattering) and on formactors or from theoretical considerations based on the large N_c limit (see section 13).)

We now show how to calculate the SU(2)xSU(2) parameters in terms of the SU(3)xSU(3) coupling constants.

Let us start with the parameters at order p^2. In the case of SU(2)xSU(2) the leading order effective Lagrangian is determined by the constants F and B, where F is the value of the pion decay constant in the limit $m_u = m_d = 0$, $m_s \neq 0$ and F^2B is the vacuum expectation value of $-\bar{u}u$ in the same limit. The value of F is easily obtained from (10.7) by setting $m_u = m_d = 0$

$$F = F_0 \left\{ 1 - \bar{\mu}_K + \frac{8\bar{M}_K^2}{F_0^2} L_4^r \right\} \tag{11.2}$$

In the same manner (9.1) leads to

$$B = B_0 \left\{ 1 - \frac{1}{3}\bar{\mu}_\eta - \frac{16\bar{M}_K^2}{F_0^2}\left(L_4^r - 2L_6^r\right) \right\} \tag{11.3}$$

(Barred quantities refer to the limit $m_u = m_d = 0$, e.g. $\bar{M}_K^2 = m_s B_0$).

To find the relations between the parameters at order p^4 it suffices to compare the generating functionals at order ϕ^4. It is convenient to compare the two expansions in powers of the external fields around the point $m_u = m_d = 0$. This is a valid procedure since the coupling constants are independent of m_u and m_d (in the case of SU(2)xSU(2) they do however depend on m_s, see (11.2, 3, 6)).

In the limit (11.1) the loop integrals involving virtual K or η mesons reduce to polynomials in the external momenta. Up to terms of order s/M_K^2 the scalar functions J^r and M^r which occur in the expression (8.13) for the unitarity corrections Z_u become constant:

$$\begin{aligned} J^r_{KK} &= -2\nu_K \\ J^r_{\eta\eta} &= -2\nu_\eta \\ J^r_{\pi\eta} &= -2\nu_\eta + \frac{1}{16\pi^2} \\ M^r_{KK} &= -\frac{1}{6}\nu_K \end{aligned} \tag{11.4}$$

where

$$\nu_P = \frac{1}{32\pi^2}\left(\ln\frac{\bar{M}_P^2}{\mu^2}+1\right) \quad ; \quad P = K, \eta \tag{11.5}$$

(Note that the functions J^r, M^r depend on the masses M_P and M_Q of the mesons running around the loop. The quantity $J^r_{\pi\eta}$ e.g. stands for the function J^r evaluated with $M_P = M_\pi$, $M_Q = M_\eta$.) The functions K and L do not contribute, because we have set $m_u = m_d$ (= 0): in this case the only unequal mass loops are loops containing a π and an η - the vertex $\hat{\Gamma}^\mu_{\pi\eta}$ is however absent, since λ_π and λ_η commute if $m_u = m_d$. Likewise, there is no contribution from $M^r_{\pi\eta}$ or from $M^r_{\eta\eta}$, because $\hat{\Gamma}^\mu_{\pi\eta} = \hat{\Gamma}^\mu_{\eta\eta} = 0$. Inserting the representations (11.4) in the expression (8.13) for the unitarity correction and adding the tree graph and tadpole contribution (8.12) one finds that the SU(3)xSU(3) expansion indeed reduces to the SU(2)xSU(2) expansion given in II, provided the renormalized coupling constants are identified as follows:

$$\begin{aligned}
\ell_1^r &= 4L_1^r + 2L_3 - \frac{1}{24}\nu_K \\
\ell_2^r &= 4L_2^r - \frac{1}{12}\nu_K \\
\ell_3^r &= -8L_4^r - 4L_5^r + 16L_6^r + 8L_8^r - \frac{1}{18}\nu_\eta \\
\ell_4^r &= 8L_4^r + 4L_5^r - \frac{1}{2}\nu_K \\
\ell_5^r &= L_{10}^r + \frac{1}{12}\nu_K \\
\ell_6^r &= -2L_9^r + \frac{1}{6}\nu_K \\
\ell_7 &= \frac{F_0^2}{8B_0 m_s}\left(1+\frac{10}{3}\bar{\mu}_\eta\right) + 4\left(L_4^r - L_6^r - 9L_7 - 3L_8^r + \frac{1}{8}\nu_K\right) \\
h_1^r &= 8L_4^r + 4L_5^r - 4L_8^r + 2H_2^r - \frac{1}{2}\nu_K \\
h_2^r &= -\frac{1}{4}L_{10}^r - \frac{1}{2}H_1^r - \frac{1}{24}\nu_K \\
h_3^r &= 4L_8^r + 2H_2^r - \frac{1}{2}\nu_K - \frac{1}{3}\nu_\eta + \frac{1}{96\pi^2}
\end{aligned} \tag{11.6}$$

As a check of the relations (11.6) one verifies that the dependence of the right- and left-hand sides on the scale μ is the same. Furthermore, one may compare the quark mass expansions of the vacuum expectation values of $\bar{u}u$ and $\bar{d}d$, of the pion mass and of the pion decay constant given in sections 9 and 10 with the corresponding SU(2)xSU(2) expansions given in II - the agreement of the two low energy representations is a rather thorough consistency check on our calculations.

We briefly discuss an application of the relation for ℓ_7. In II we have shown that this constant determines the mass difference

$$(M^2_{\pi^+} - M^2_{\pi^0})_{QCD} = (m_u - m_d)^2 \frac{2B^2}{F^2} \ell_7 \{1 + O(\hat{m})\} \tag{11.7}$$

Inserting the expressions for F, B and ℓ_7 given above, this becomes

$$(M^2_{\pi^+} - M^2_{\pi^0})_{QCD} = \frac{1}{4} \frac{(m_u - m_d)^2}{(m_s - \hat{m})} B_0 \{1 + \frac{8}{3}\bar{\mu}_\eta + 2\bar{\mu}_K \tag{11.8}$$

$$+ \frac{32}{F_0^2} \bar{M}_K^2 (-\tfrac{1}{2}L_4^r + L_6^r - 9L_7 - 3L_8^r + \tfrac{1}{8}\nu_K) + O(\hat{m}, m_s^2)\}$$

Note that contributions of relative order $\hat{m}$ are neglected - in principle these contributions could also be extracted from our low energy representation. We are shortcutting a tedious calculation here at the loss of some information. A representation for all nonanalytic terms in the quark mass expansion of $M^2_{\pi^+} - M^2_{\pi^0}$ up to and including $M^2 \log M$ is given in Appendix C of I. (Indeed one easily checks that the chiral logarithms in (11.8) agree with (C.7)). The information contained in (11.8) goes beyond leading logarithms: the scale is fixed in terms of the coupling constants L_i^r. We eliminate these constants in favour of measured quantities and rewrite (11.8) in the form

$$(M^2_{\pi^+} - M^2_{\pi^0})_{QCD} = \frac{(M^2_{K^+} - M^2_{K^0})^2_{QCD}}{3(M^2_\eta - M^2_\pi)} \{1 + \frac{8}{3}\Delta_{GMO} + \frac{M_K^2}{8\pi^2 F_0^2}(1 + 6\ln\frac{M_K^2}{M_\eta^2})$$

$$+ O(\hat{m}, m_s^2)\} \tag{11.9}$$

At leading order in the quark mass expansion the mass difference is exclusively due to $\pi^0\eta$ mixing. At first nonleading order other effects such as mixing with the η' contribute. The curly bracket in (11.9) accounts for these corrections in a parameter free manner. As was anticipated in I the correction to the $\pi^0\eta$

mixing formula is large, of order 50%. It turns out that the higher order contributions amplify the perturbation generated by $m_u - m_d$. Numerically,

$$(M_{\pi^+} - M_{\pi^0})_{QCD} = 0.17 \pm 0.03 \text{ MeV} \tag{11.10}$$

where the error includes the uncertainties in the value of $(M_{K^0} - M_{K^+})_{QCD}$ as well as the uncertainties due to terms of $O(\hat{m}, m_s^2)$. This leaves

$$(M_{\pi^+} - M_{\pi^0})_{QED} = 4.43 \pm 0.03 \text{ MeV} \tag{11.11}$$

for the electromagnetic contributions.

12. The η'

In the systematic low energy expansion the η' does not play a special role as compared to other excited states such as the ρ. The presence of these states only manifests itself indirectly in the values of the low energy constants F_0, B_0, L_1,.... Both beacuse the η' is crucial for an understanding of the large N_c limit (Witten, 1979; Veneziano 1979; Di Vecchia 1979; Rosenzweig, Schechter and Trahern 1980; Di Vecchia and Veneziano 1980; Kawarabayashi and Ohta 1980; Nath and Arnowitt 1981) and because $\eta\eta'$ mixing is known to affect the low energy properties of the η in a significant way we briefly consider an extension of our framework which explicitly includes the η' degrees of freedom. It suffices to drop the constraint det $U = \exp(-i\theta)$; the unitary matrix $U(x)$ then contains nine fields

$$U(x) = \exp\{\tfrac{i}{3}\varphi_0(x)\}\exp\{i\varphi(x)\}$$

where $\phi(x)$ is hermitean and traceless and where the single component field $\phi_0(x)$ is related to the determinant of U : det $U = \exp(i\phi_0)$. In the limit of exact SU(3), $m_u = m_d = m_s$, the field $\phi(x)$ describes the eight Goldstone bosons and $\phi_0(x)$ is the η' field.

It is straightforward to adapt the analysis given in section 3 to this more general setting. We first note that the quantity $\phi_0(x) + \theta(x)$ is invariant under chiral transformations. Chiral symmetry does therefore not imply that the most general Lagrangian of order p^0 is a constant, but only implies that it is a function of $\phi_0 + \theta$

$$\mathcal{L} = -V_0(\varphi_0 + \Theta) + \cdots$$

To order p^2 the general effective Lagrangian consistent with Lorentz invariance and with chiral symmetry now takes the form

$$\mathcal{L}_1 = -V_0 + V_1 \mathrm{tr}\, \nabla_\mu U^+ \nabla^\mu U + V_2 \mathrm{tr}(s-ip)U + V_2^* \mathrm{tr}(s+ip)U^+$$
$$+ V_3 \nabla_\mu \varphi_0 \nabla^\mu \Theta + V_4 \nabla_\mu \Theta \nabla^\mu \Theta + V_5 \nabla_\mu \varphi_0 \nabla^\mu \varphi_0 \qquad (12.1)$$

where $\nabla_\mu \phi_0$ denotes the covariant derivative

$$\nabla_\mu \varphi_0 = \partial_\mu \varphi_0 - 2\,\mathrm{tr}\, a_\mu = -i\, \mathrm{tr}(U^+ \nabla_\mu U) \qquad (12.2)$$

and where the potentials V_i depend on $\phi_0 + \theta$ only:

$$V_i = V_i(\varphi_0 + \Theta) \qquad (12.3)$$

The potential V_5 may be eliminated by a change of variables of the type $U \to U \exp\{if(\phi_0 + \theta)\}$; we therefore set $V_5 = 0$. (For the Lagrangian to be real, V_0, V_1, V_3 and V_4 must be real, whereas V_2 may be complex. Furthermore, invariance under parity implies $V_i(\alpha) = V_i^*(-\alpha)$.)

To discuss the low energy information contained in the Lagrangian (12.1) we again need to work out the minimum of the Euclidean action. To leading order in the low energy expansion the field $\phi_0(x)$ sits at the minimum of the potential V_0. In order for the ground state to be an eigenstate of parity if $\theta_0 = 0$ this minimum must occur at $U_0 = U_{0_0}^+$. This implies that $\exp(+i\phi_0) = \pm 1$ i.e. $\phi_0 = 0$ or $\phi_0 = \pi$. If the minimum is at $\phi_0 = \pi$ we change variables, replacing U by $-$ U. In the new variables the minimum occurs at $\phi_0(x) = 0$ if θ vanishes and at $\phi_0(x) = -\theta(x)$ if the external field $\theta(x)$ is not switched off (but is close to $\theta(x) = 0$, see section 4).

To determine the generating functional to second order in the momenta we expand the potentials V_i in a power series of $\phi_0 + \theta$. This power series contains an infinite sequence of vertices, describing processes which involve an increasing number of η'-mesons. (In contrast to the couplings of the Goldstone bosons among themselves, which at order p^2 are fixed by the two constants F_0 and B_0, chiral symmetry does not restrict the couplings of the η' very strongly - the general solution of the Ward identities contains infinitely many coupling constants.) To determine the mass spectrum and the one particle matrix elements of the currents we need only consider those vertices which are at most quadratic in the meson fields. Expanding the general Lagrangian (12.1) in powers of ϕ and ϕ_0 one finds that the masses of the charged mesons are not affected by the presence of the η', but π^0, η and η' mix. If we put $m_u = m_d = \hat{m}$ there is no mixing with the π^0 and the $\eta\eta'$ mixing angle δ becomes

$$
\begin{aligned}
F_0 \varphi_8 &= \eta \cos\delta + \eta' \sin\delta \\
\frac{F_0}{\sqrt{6}} \varphi_0 &= -\eta \sin\delta + \eta' \cos\delta \\
\mathrm{tg}\, 2\delta &= -\tfrac{4}{3}\sqrt{2}\,\gamma\,(\mathring{M}_K^2 - \mathring{M}_\pi^2)/(M_0^2 - \mathring{M}_\eta^2)
\end{aligned}
\tag{12.4}
$$

where the constants F_0, γ and M_0 are given by

$$
\begin{aligned}
F_0^2 &= 4 V_1(0) \\
F_0^2 B_0 &= 2 V_2(0) \\
F_0^2 M_0^2 &= 6 V_0''(0) + \tfrac{4}{3} \mathrm{tr}\, \mathcal{M} \{ V_2(0) + 6i V_2'(0) - 9 V_2''(0) \} \\
\gamma &= 1 + 3i \frac{V_2'(0)}{V_2(0)}
\end{aligned}
\tag{12.5}
$$

and the masses $\mathring{M}_\pi$, $\mathring{M}_k$, $\mathring{M}_\eta$ stand for

$$
\begin{aligned}
\mathring{M}_\pi^2 &= 2\hat{m} B_0 \\
\mathring{M}_K^2 &= (\hat{m} + m_s) B_0 \\
\mathring{M}_\eta^2 &= \tfrac{2}{3}(\hat{m} + 2m_s) B_0
\end{aligned}
\tag{12.6}
$$

Denoting the eigenvalues of the mass operator by M_π, M_K, M_η, $M_{\eta'}$ one finds

$$\begin{aligned} M_\pi &= \overset{\circ}{M}_\pi \quad , \quad M_K = \overset{\circ}{M}_K \\ M^2_{\eta'} + M^2_\eta &= M^2_0 + \overset{\circ}{M}{}^2_\eta \\ M^2_{\eta'} - M^2_\eta &= (\cos 2\delta)^{-1} (M^2_0 - \overset{\circ}{M}{}^2_\eta) \end{aligned} \tag{12.7}$$

The mixing angle may therefore also be expressed in terms of the eigenvalues of the mass operator:

$$\sin^2 \delta = \frac{1}{3} \frac{(4M^2_K - M^2_\pi - 3M^2_\eta)}{(M^2_{\eta'} - M^2_\eta)} \tag{12.8}$$

If one assumes (see however the discussion below) that the mixing with the η' is the main reason for the small difference between the observed mass of the η (M_η = 548.8 MeV) and the mass which one obtains from M_π, M_K with the Gell-Mann-Okubo formula ($\overset{\circ}{M}_\eta$ = 566 MeV), then the mixing angle and M_0 become

$$\delta = -10.2^\circ \quad , \quad M_0 = 947 \text{ MeV} \tag{12.9}$$

(to be compared with $M_{\eta'}$ = 957.6 MeV). Furthermore, (12.4) implies

$$\gamma = 0.5 \tag{12.10}$$

The coupling constants of the eighth component of the axial current to η and η' become

$$F^8_\eta = F_0 \cos\delta \quad ; \quad F^8_{\eta'} = F_0 \sin\delta \tag{12.11}$$

whereas the couplings to the singlet current ($\lambda^0 = \sqrt{\frac{2}{3}}\,\mathbb{1}$) are

$$\begin{aligned} F^0_\eta &= -F' \sin\delta \quad ; \quad F^0_{\eta'} = F' \cos\delta \\ F' &= F_0 \left\{ 1 - \frac{6V_3(0)}{F^2_0} \right\} \end{aligned} \tag{12.12}$$

Finally, the matrix elements of the winding number density are given by

$$\omega = \frac{1}{16\pi^2} \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu}$$

$$\langle 0|\omega|\eta\rangle = -\frac{\sin\delta}{\sqrt{6}} \left\{ A + M_\eta^2 \left(F' - \frac{F_0}{\gamma} \right) \right\} \tag{12.13}$$

$$\langle 0|\omega|\eta'\rangle = \frac{\cos\delta}{\sqrt{6}} \left\{ A + M_{\eta'}^2 \left(F' - \frac{F_0}{\gamma} \right) \right\}$$

$$A = \frac{F_0}{\gamma} \left\{ M_0^2 - \frac{1}{3} \gamma^2 (2M_K^2 + M_\pi^2) \right\}$$

Apart from F_0, M_0 and γ, the one particle matrix elements thus involve a single unknown F' related to $V_3(0)$.

One may proceed to work out the couplings of the η' to the Goldstone bosons (Di Vecchia, Nicodemi, Pettorino and Veneziano 1981). In the limit $m_u = m_d = 0$ the effective Lagrangian describing the decay $\eta' \to \eta\pi\pi$ e.g. is given by

$$\mathcal{L} = -\frac{6}{F_0^4} V_1''(0) \sin 2\delta\, \eta\eta' \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} \tag{12.14}$$

Indeed, at order p^2 this is the most general Lagrangian consistent with chiral symmetry. Since $V_1''(0)$ is an unknown constant, the total rate is not fixed by chiral symmetry, but the distribution over the Dalitz plot is determined. Acutally, the Dalitz plot slope which follows from (12.14) disagrees with observation. As pointed out by Singh and Dasupathy (1975) and Despande and Truong (1978) the SU(2)xSU(2) low energy theorem for the slope parameter fails in this case because the decay spectrum is distorted by the proximity of scalar resonances. We do not discuss the matter further here, but refer the reader to the literature on the subject.

What we instead wish to discuss is the validity of the assumption underlying the value $\delta \simeq -10^0$ for the $\eta\eta'$ mixing angle. As emphasized above this value only follows if one assumes that the η' is the main source of the observed small deviation from the Gell-Mann-Okubo formula. In section 10 we have given a general expression for this deviation, based on the low energy expansion to order p^4. This expression does not involve the assumption that the discrepancy Δ_{GMO} is dominated by the η'. To relate the effective η' Lagrangian discussed in the present section to the general framework we note that in the region of small momenta and small quark masses

$$p^2, M_\eta^2 \ll M_{\eta'}^2 \qquad (12.15)$$

the propagator describing η'-exchange reduces to a constant

$$(M_{\eta'}^2 - p^2)^{-1} \longrightarrow (M_{\eta'}^2)^{-1}$$

If we restrict ourselves to $\theta = \mathrm{tr}\, a_\mu = 0$ then the contribution from η'-exchange reduces to the following local effective Lagrangian of order p^4

$$\mathcal{L}^{\eta'} = -\frac{\gamma^2}{48}\frac{F_0^2}{M_{\eta'}^2}\left[\mathrm{tr}(\chi U^+ - U\chi^+)\right]^2 + O(p^6) \qquad (12.16)$$

In the general low energy expansion the η' thus only shows up in the form of a contribution to the low energy constant L_7

$$L_7^{\eta'} = -\frac{\gamma^2}{48}\frac{F_0^2}{M_{\eta'}^2} \qquad (12.17)$$

In connection with η'-dominance the natural question to ask, therefore, is whether the contribution from the η' dominates the constant L_7. To answer this question we first note that L_7 is not renormalized by meson loops ($\Gamma_7 = 0$, $L_7 = L_7^r$). Furthermore, one may verify that the exchange of scalar or vector mesons does not contribute to L_7. The exchange of the pseudoscalar octet containing the π' does contribute, however, with a sign opposite to the contribtuion from the η'. It therefore appears to be reasonable to assume the η' contribution to be mainly responsible for the value of L_7, provided this constant turns out to be negative. (At any rate the η' dominates in the large N_c limit.)

The value of the constant L_7 may be extracted from measured quantities as follows. In section 10 we have introduced three asymmetry parameters Δ_{GMO}, Δ_F and Δ_M which are measured ($\Delta_{GMO} = 0.21$, $\Delta_F = 0.22$, $\Delta_M = \pm\, 0.09$) and which (at the order in the low energy expansion we are considering) are determined by the constants L_5^r, L_7 and L_8^r. It therefore suffices to solve the three relations (10.10, 11, 12) for L_5^r, L_7 and L_8^r. The result for L_7 is

$$L_7 = -(0.4 \pm 0.15) \cdot 10^{-3} \tag{12.18}$$

The sign is negative, as it is required if the η' is the main source of L_7. Using (12.17) to determine the value of the coupling constant γ which measures the strength of $\eta\eta'$ mixing (see (12.4)) we get $\gamma \simeq 1.4$. Since (12.17) only holds to lowest order in m_s/M^2_η, and does therefore not distinguish M^2_η, from $M^2_{\eta'} - M^2_\eta$, the value obtained in this manner for γ only represents a rough estimate. To obtain a reliable value for γ and for the $\eta\eta'$ mixing angle δ we note that the crucial quantity is the amount by which $\eta\eta'$ mixing drives the $(\mathrm{mass})^2$ of the η away from the value $\frac{1}{3}(4\, M^2_K - M^2_\pi)$ predicted by the Gell-Mann-Okubo formula. In lowest order of the quark mass expansion the masses are given by (12.6). The Zweig rule implies that the mass of the pion is affected only little by effects of order M^2 (see section 13). With $m_s : \hat{m} = 25$ we thus have $\overset{o}{M}_\pi \simeq 135$ MeV, $\overset{o}{M}_K \simeq 487$ MeV, $\overset{o}{M}_\eta \simeq 557$ MeV. The overall effect of the corrections of order M^2 (which shift these values into the observed masses) is therefore quite small ($\lesssim 10$ MeV). To determine the mixing angle we however need to split the second order contributions to M_η into a piece from L_7 (which is blamed on $\eta\eta'$ mixing) and a remainder, due to chiral logarithms and to the low energy constants L_4, L_5, L_6 and L_8 (see (10.7)). With (12.18) one finds that L_7 shifts M_η downward by 80 ± 30 MeV (the remainder shifts it in the opposite direction by almost the same amount). The mixing angle required by (12.8) for such a large shift is

$$\delta = -20^\circ \pm 4^\circ \tag{12.19}$$

The corresponding value of the coupling constant γ may then be determined from (12.4) and (12.7) with the result

$$\gamma = 1.1 \pm 0.3 \tag{12.20}$$

Note that the values obtained for the strength γ of the $\eta\eta'$ coupling and for the mixing angle δ are twice as large as the canonical values $\gamma = 0.5$, $\delta = -10^0$, extracted directly from the Gell-Mann-Okubo formula: the canonical values underestimate $\eta\eta'$ mixing by about a factor two. (We emphasize that this statement holds within the specific framework used here, which includes all effects of order M^2. If one only takes $\eta\eta'$ mixing into account and neglects other effects of the same order, one may just as well stick to the canonical values.)

13. Large N_c, Zweig rule

If the number of colours is sent to infinity and the coupling constant g is sent to zero in such a manner that the product $N_c\, g^2$ stays constant, the Green's functions of the theory are proportional to a power of N_c (t'Hooft 1974; Veneziano 1976; Witten 1979, 1980). We denote the general connected Green's function containing Q quark currents and W winding number densities by

$$G_{QW} = \langle 0|T\, j_1(x_1)\ldots j_Q(x_Q)\,\omega(y_1)\ldots\omega(y_W)|0\rangle_{\text{connected}}$$
$$j_i = \bar{q}\,\Gamma_i\, q \tag{13.1}$$
$$\omega = (16\pi^2)^{-1}\, \mathrm{tr}\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

where the colour neutral matrices Γ_i act on the spin and on the flavour of the quarks (note that our normalization of $G_{\mu\nu}$ is $\mathcal{L} = -\,\mathrm{tr}\, G_{\mu\nu}G^{\mu\nu}/(2\, g^2) + \ldots$). For large N_c this Green's function is of order

$$G_{QW} = \begin{cases} O(N_c^{2-W}) & Q = 0 \\ O(N_c^{1-W}) & Q \neq 0 \end{cases} \tag{13.2}$$

(Note that this counting rule only holds for generic momenta. The exchange of an η' e.g. generates a pole factor $(M^2_{\eta'} - p^2)^{-1}$ which at p = 0 produces an additional power of N_c, see below.) The leading contributions to Green's functions containing quark currents (Q ≠ 0) arise from graphs with a single quark loop (planar graphs with the quark loop running at the edge of the diagram). These graphs are given by the functional integral over the gluon field of a product of the form $\mathrm{tr}(\Gamma_{i_1} S\Gamma_{i_2}S\ldots\Gamma_{i_Q}S)$ where $i_1,\ldots, i_Q$ is some permutation of 1,..., Q and where S denotes the quark propagator in the presence of the gluon field. If the quark masses are set equal to zero, the propagator becomes flavour independent. In the chiral limit the leading contribution to G_{QW} therefore depends on the flavour of the currents only through the trace $\mathrm{tr}(\lambda_{i_1},\ldots\lambda_{i_Q})$ where λ_i is the flavour factor in the matrix Γ_i.

The large N_c behaviour of the generating functional is easily obtained from (13.2)

$$Z(v,a,s,p,\theta) = N_c^2 f_0\left(\frac{\theta}{N_c}\right) + N_c f_1\left(v,a,s,p,\frac{\theta}{N_c}\right) + O(1) \tag{13.3}$$

where the functionals $f_0(\alpha)$ and $f_1(v,a,s,p,\alpha)$ are independent of N_c.

The counting rules for the one particle matrix elements are (Witten 1979)

$$\begin{aligned} \langle 0|j|\text{meson}\rangle &= O(N_c^{1/2}) \; ; \; \langle 0|j|\text{glueball}\rangle = O(1) \\ \langle 0|\omega|\text{meson}\rangle &= O(N_c^{-1/2}) \; ; \; \langle 0|\omega|\text{glueball}\rangle = O(1) \end{aligned} \tag{13.4}$$

To analyze the large N_c behaviour of the effective Lagrangian (12.1) it suffices to expand the matrix U in terms of the meson fields

$$U = 1 + i\,\frac{\varphi_0}{3} + i\,\varphi + \ldots \tag{13.5}$$

and to look at the terms which are independent of ϕ_0 and ϕ. Comparing these terms with (13.3) we obtain ($\alpha = \phi_0 + \theta$)

$$\begin{aligned} V_0(\alpha) &= N_c^2\, v_0(\alpha/N_c) \\ V_1(\alpha) &= N_c\, v_1(\alpha/N_c) \\ V_2(\alpha) &= N_c\, v_2(\alpha/N_c) \\ V_3(\alpha) &= v_3(\alpha/N_c) \\ V_4(\alpha) &= v_4(\alpha/N_c) \end{aligned} \tag{13.6}$$

where the functions $v_i(x)$ are independent of N_c. (Formally, these properties of the potentials are in conflict with periodicity. If the potentials $V_i(\phi_0 + \theta)$ are periodic with period 2π then the representations (13.6) can hold only in a neighbourhood of the origin (Witten 1980; we are indepted to G. Veneziano for a discussion of this problem).) The relations (12.5, 12, 13) then show that the constants F_0, B_0, F', γ, M_0 and A have the following large N_c behaviour

$$
\begin{aligned}
F_0 &= O(N_c^{1/2}) \\
B_0 &= O(1) \\
F' &= F_0 \{1 + O(1/N_c)\} \\
\gamma &= 1 + O(1/N_c) \\
M_0^2 &= \left\{\frac{1}{N_c}\frac{3}{2}\frac{v_0''(0)}{v_1(0)} + \frac{2}{3}B_0 \operatorname{tr} \mathcal{M}\right\}\{1 + O(1/N_c)\} \\
A &= O(N_c^{-1/2})
\end{aligned}
\tag{13.7}
$$

in agreement with the rules (13.4) for the one particle matrix elements. The coupling constant F' tends to F_0, the quantity γ which determines $\eta\eta'$ mixing tends to 1. In the chiral limit $M^2_{\eta'}$ is of order $1/N_c$. For nonvanishing quark masses the mass formulae (12.4, 7) show that in the formal limit $N_c \to \infty$ the η' will only contain strange quarks with $M^2_{\eta'} = 2\, m_s B_0$ whereas the η will only contain u and d quarks and will be degenerate with the pion.

The prediction $\gamma = 1$ is in good agreement with the value $\gamma = 1.1 \pm 0.3$ required by the mass spectrum: the large N_c limit thus also predicts that the η mixes more strongly with the η' than indicated by the canonical value $\gamma = 0.5$ (the mixing angle which follows from (12.4, 6, 7) with $\gamma = 1$ and with the observed masses M_π, M_K, $M_{\eta'}$ is $\delta = -20^0$, see also Kawarabayashi and Ohta 1980). In order to understand why the value $\gamma = 1$ does not imply a strong distortion of the Gell-Mann-Okubo formula, one however needs to take <u>all</u> contributions of order M^2 to the meson mass formulae into account - the large N_c prediction for the η mass based on the tree Lagrangian alone would be $M_\eta \simeq M_K$!

A different test of the large N_c predictions is provided by the decays $\eta \to 2\gamma$, $\eta' \to 2\gamma$ which are sensitive to $\eta\eta'$-mixing (Gilman 1979; Chanowitz 1980; Minkowski 1982; Roiesnel and Truong 1982; Field 1983). The lowest order contribution to these decays however originates in the anomalous piece Z_A of the generating functional which is of order p^4. To reliably calculate SU(3) breaking effects we would have to carry the low energy expansion to order p^6. This is beyond our scope.

Finally, let us work out the large N_c behaviour of the low energy constants which appear in the general effective Lagrangian (6.16). This Lagrangian is relevant for the behaviour of the Green's functions at momenta small in comparison to $M_{\eta'}$. As pointed out above the large N_c counting rules do not apply directly in this case, because the exchange of an η' (not visible in the effective Lagrangian, but implicitly present in the low energy constants) generates a factor $1/M^2_{\eta'}$, which upsets the power counting. To extend the validity of the counting rules to $p^2 \ll M^2_{\eta'}$, we note that small denominators occur only if the Green's function splits into two or more pieces connected by the exchange of one or several η' mesons (one particle reducible contributions). The general structure of the η' couplings permitted by chiral symmetry is given by the Lagrangian (12.1) up to and including terms of order p^2 (put θ = tr $a_\mu = 0$). Since this Lagrangian does not contain any terms coupling the η' to the other fields at order p^0 we conclude that the exchange of a single η' leads to a contribution of order p^4; with two consecutive exchanges we obtain a contribution of order p^6 etc. To obtain the low energy expansion up to and including p^4 at most one exchange of an η' can contribute. In fact, this contribution was worked out in section 12 with the result that all constants in the effective Lagrangian except L_7 are unaffected. The contribution to L_7 is of order N_c^2 and is given explicitly in (12.17). The large N_c counting rules therefore apply to all terms in the Lagrangian (6.16) except to L_7:

$$L_i = O(N_c) \;, \; i \neq 7 \;; \; H_K = O(N_c)$$

(The leading term in L_4 and L_6 happens to vanish, see below.)
As mentioned above, in the chiral limit the leading contribution to the Green's functions involving quark currents are proportional to traces over the product of the relevant flavour matrices. To evaluate this property we again look at the terms in the effective Lagrangian which are independent of the pion field, i.e. put $U = \mathbb{1}$. The contribution from L_6, L_8 and H_2 then reduces to

$$4B_0^2 \left\{ L_6 (\mathrm{tr}\, s)^2 + (H_2 + 2L_8)\, \mathrm{tr}(s^2) + (H_2 - 2L_8)\, \mathrm{tr}(p^2) \right\}$$

The vertices generated by L_8 and H_2 are indeed of the required form, but the contribution from L_6 is of a different structure. Hence L_6 is sppressed by at least one power of N_c. Analogously, one concludes that L_4 is also suppressed. Concerning the constants L_1, L_2 and L_3 one has to keep in mind that the algebraic identitiy (7.24) allows one to express products of traces in terms of traces of products. This leads to the conclusion that only the combination 2 $L_1 - L_2$ is suppressed.

We thus have the following large N_c behaviour

$$\begin{aligned} &O(N_c^2) : L_7 \\ &O(N_c) : L_1, L_2, L_3, L_5, L_8, L_9, L_{10}, H_1, H_2 \qquad (13.8) \\ &O(1) : 2L_1 - L_2, L_4, L_6 \end{aligned}$$

The suppression of L_6 guarantees that the vacuum expectation value $<0|\bar{u}u|0>$ is not sensitive to the mass of the strange quark (see (9.1)). The suppression of L_4 implies that the value of F_π is also not strongly affected by m_s (cf. 10.7). Finally, as can be seen from (10.7) the contribution to M_π^2 proportional to m_s only involves L_4 and L_6 and is therefore also suppressed. The large N_c limit thus gives a theoretical basis for the Zweig rule ('t Hooft 1974; Veneziano 1976; Witten 1979). We emphasize that the suppression of the constants $2L_1 - L_2$, L_4 and L_6 is a general feature independent of the chiral structure, in contrast to the large N_c enhancement of L_7 which holds only to the extent that $M_{\eta'}$ is small in comparison to the scale of the theory.

The one loop contributions to the generating functional are of order 1, i.e. are suppressed by one power of N_c in comparison with the leading graphs. Shifting the renormalization point in the coupling constant L_4 e.g. changes the value of this quantity by a contribution of order 1, in accord with the counting rules (13.8).

14. Values of the low energy constants

We are now in a position to estimate the values of the low energy constants $L_1, \ldots, L_{10}$. We have shown in II that the experimental information on the D-wave $\pi\pi$ scattering lengths, on the electromagnetic charge radius of the pion and on the decay $\pi \to e\nu\gamma$ allows one to determine four of the seven low energy constants of SU(2)xSU(2): ℓ_1, ℓ_2, ℓ_5 and ℓ_6. Using the formulae (11.6) which relate these constants to the low energy expansion of SU(3)xSU(3), this information may be taken over to determine four of the ten low energy constants $L_1, \ldots, L_{10}$. The measured value of $F_K : F_\pi$, the observed deviation from the Gell-Mann-Okubo formula and the relation (10.18) between $(M^2_{K^0} - M^2_{K^+}) : (M^2_K - M^2_\pi)$ and the quark mass ratio $(m_d - m_u) : (m_s - \hat{m})$ (on which we have independent information from isospin breaking in the baryon spectrum and from $\rho\omega$ mixing) provide three further constraints on these low energy constants. Among the three remaining unknowns, one (L_1) could presumably be measured in K_{ℓ_4} decay or in $K\pi$ scattering. The other two (L_4 and L_6) may be more difficult to measure directly; they determine the amount by which the strange quark mass affects the values of F_π and of M_π. We use large N_c arguments (Zweig rule) to estimate the three constants on which we do not have direct experimental information and will point out some tests of the predictions obtained in this manner.

Let us start with the constant L_2. According to (11.6) this quantity is related to the SU(2)xSU(2) low energy constant ℓ_2 which can be measured through the D-wave $\pi\pi$ scattering lengths

$$\ell_2^r = 10\pi F_\pi^4 (a_2^0 - a_2^2) + \frac{1}{48\pi^2}\left\{ \ln \frac{M_\pi^2}{\mu^2} + \frac{27}{20} \right\} \tag{14.1}$$

For definiteness we evaluate all coupling constants at scale

$$\mu = M_\eta$$

The observed scattering lengths as reported by Petersen in the compilation of coupling constants and low energy parameters (Nagels et al. 1979)

$$\begin{aligned} a_2^0 &= (17 \pm 3) \cdot 10^{-4} \\ a_2^2 &= (1.3 \pm 3) \cdot 10^{-4} \end{aligned} \tag{14.2}$$

(in units of M_{π^+}) lead to

$$\ell_2^r = (6.8 \pm 2.7) \cdot 10^{-3} \tag{14.3}$$

Using the relation (11.6) this value may be converted into a value for L_2^r; the result is shown in the table.

The D-wave scattering lengths also allow us to pin down the combination

$$L_3 + (2L_1^r - L_2^r) = \frac{1}{2}\ell_1^r - \frac{1}{4}\ell_2^r$$
$$= \frac{5}{2}\pi F_\pi^4(-2a_2^0 + 5a_2^2) - \frac{1}{1536\pi^2} \qquad (14.4)$$

of SU(3)xSU(3) low energy constants. Inserting the measured values (14.2) this becomes

$$L_3 + (2L_1^r - L_2^r) = (-4.4 \pm 2.5)\cdot 10^{-3} \qquad (14.5)$$

The constants L_5^r, L_7 and L_8^r may be determined from the measured values of the asymmetry parameters Δ_{GMO}, Δ_F and Δ_M in the manner discussed in section 12. The results quoted in the table include a rough estimate of the uncertainties due to higher order effects.

The constant L_9 is related to the SU(2)xSU(2) parameter ℓ_6 which fixes the electromagnetic charge radius of the pion

$$\ell_6^r = -\frac{1}{6}F_\pi^2 \langle r^2\rangle_V^\pi - \frac{1}{96\pi^2}\left(\ln\frac{M_\pi^2}{\mu^2} + 1\right) \qquad (14.6)$$

With the value (Dally et al. 1982)

$$\langle r^2\rangle_V^\pi = 0.439 \pm 0.03 \, fm^2 \qquad (14.7)$$

we obtain

$$\ell_6^r = -(14.5 \pm 1.1)\cdot 10^{-3} \qquad (14.8)$$

The relation (11.6) then leads to the number for L_9^r contained in the table.

As was shown in II the value of the constant ℓ_5 may be extracted from data on the structure term in the decay $\pi \to e\nu\gamma$. The value $\bar{\ell}_5 = 13.9 \pm 1.3$ quoted in II is equivalent to

$$\ell_5^r = -(5.9 \pm 0.7)\cdot 10^{-3} \qquad (14.9)$$

The table shows the corresponding value of L_{10}^r, obtained from (11.6).

We now have pinned down all SU(3)xSU(3) low energy constants except three: L_1 (which together with the known constant L_2 determines L_3 through (14.5)), L_4 and L_6. To estimate these constants we invoke the Zweig rule: at large N_c the combination $L_1 - \frac{1}{2} L_2$, L_4 and L_6 are suppressed. The Zweig rule thus in particular predicts that L_1^r is close to $\frac{1}{2} L_2^r$ (note that the difference $L_1^r - \frac{1}{2} L_2^r$ is independent of the renormalization scale μ). The experimental uncertainty in the value of L_2^r is of order 30%. We consider it very unlikely that the deviations from the Zweig rule could be of comparable size. For this reason the value quoted for the constant L_1^r in the table is the number $\frac{1}{2} L_2^r$. Since $2 L_1^r - L_2^r$ is small the constant L_3 must be close to the value of the r.h.s. in (14.5). Again the uncertainty due to the Zweig rule violating contribution is expected to be small in comparison with the experimental error in (14.5). It should be possible to test the Zweig rule predictions for L_1^r and L_3 in K_{ℓ_4} decay or in $K\pi$ scattering.

The value of the constants L_4^r and L_6^r depends on the renormalization point μ: the meson loops violate the Zweig rule (the large N_c limit forbids contributions of order N_c to L_4^r and to L_6^r, the loops only contribute at order 1). To estimate the size of the loop contributions we consider the change in the value of the constants L_4^r, L_6^r produced by a change of the renormalization scale μ by a factor of two. The change is given by

$$\Delta L_i^r = L_i^r(\mu_2) - L_i^r(\mu_1) = \frac{\Gamma_i}{16\pi^2} \ln \frac{\mu_1}{\mu_2} \tag{14.10}$$

With $\mu_1 : \mu_2 = 2$ this gives

$$\Delta L_4^r = 0.5 \cdot 10^{-3} \ , \ \Delta L_6^r = 0.3 \cdot 10^{-3} \tag{14.11}$$

If the main contribution to L_4 and L_6 arises from the loops then one expects L_4^r, L_6^r, normalized at a scale μ of order 500 MeV or 1 GeV to be of order ΔL_4^r, ΔL_6^r respectively. To substantiate this crude estimate we look at the sensitivity of the pion decay constant to the mass of the strange quark: F_π should be less sensitive to m_s than F_K. The relation (10.7) shows that the ratio $F_\pi : \hat{F}_\pi$ (where $\hat{F}_\pi$ is the value of F_π in the limit $m_s = 0$) is given by

$$\frac{F_\pi}{\hat{F}_\pi} = 1 - \mu_K + \hat{\mu}_K + \frac{8 B_0 m_s}{F_0^2} L_4^r \tag{14.12}$$

The renormalization point dependence of the constant L_4^r is cancelled by the chiral logarithms appearing in this formula. At the scale $\mu = M_\eta$ the quantity $\mu_K - \hat{\mu}_K$ is less than 1%; the value of L_4^r at this scale is therefore a good measure for the Zweig rule violating contributions to F_π. Denoting the value of F_K in the limit $m_s = 0$ by $\hat{F}_K$ the low energy representation (10.7) gives

$$\frac{F_K}{\hat{F}_K} = 1 - \frac{3}{2}(\mu_K - \hat{\mu}_K) - \frac{3}{4}(\mu_\eta - \hat{\mu}_\eta) + \frac{8B_0 m_s}{F_0^2}\left(L_4^r + \frac{1}{2}L_5^r\right) \tag{14.13}$$

Again, at the scale $\mu = M_\eta$ the chiral logarithms amount to less than 1%. In order for F_π to be less sensitive to m_s than F_K we must have

$$|L_4^r| \ll \frac{1}{2}|L_5^r| \tag{14.14}$$

i.e. the reference value for the size of the Zweig rule violating quantity L_4^r is $\frac{1}{2} L_5^r = 10^{-3}$. In view of this we consider the estimate for L_4^r given in the table as conservative. (Expressed in terms of L_4^r the preliminary value $\bar{\ell}_4 = 4.6 \pm 0.9$ given in II is equivalent to $L_4^r = (0.5 \pm 0.7) \cdot 10^{-3}$).

The size of L_6^r may be estimated either by investigating the dependence of the vacuum expectation value $\langle 0|\bar{u}u|0\rangle$ on m_s or by calculating the amount by which the pion mass changes if m_s is varied. The estimate for L_6^r given in the table implies that the change in $\langle 0|\bar{u}u|0\rangle$ as m_s varies from $m_s = 0$ to its physical value is less than 25%, the corresponding change in M_π^2 (due to both L_4^r and L_6^r) could still be as large as 40%. (The value of the SU(2)xSU(2) low energy constant $\bar{\ell}_3 = 2.9 \pm 2.4$ given in the above reference is equivalent to the estimate $L_4^r - 2 L_6^r = \pm 0.5 \cdot 10^{-3}$. This bound guarantees that the value of M_π^2 in a world with $m_s = 0$ differs from the physical value by less than 20%).

To some extent these attempts at quantifying the Zweig rule are arbitrary - the fact that the rule holds to an amazing degree of accuracy e.g. in the mass spectrum (ρ-ω degeneracy, δ - S^* degeneracy etc.) does not necessarily imply that the large N_c arguments can be trusted in the context of the matrix elements we are discussing here. (A rough test of the Zweig rule prediction for L_4^r is described in the reference quoted above: the value of the scalar radius of the pion extracted from $\pi\pi$ scattering is consistent with the value one predicts on the basis of the Zweig rule. Since the experimental uncertainties are however rather large, only a large deviation from the Zweig rule is excluded by this test.)

15. Summary

We have analyzed the structure of the functional Z which generates the Green's functions of the quark currents and of the operator tr $G_{\mu\nu}\tilde{G}^{\mu\nu}$. The Ward identities associated with chiral $U(N_f)\times U(N_f)$ specify the manner in which this functional transforms under local chiral transformations of the external fields. If the quark loops did not generate anomalies, Z would be invariant under these transformations. The anomalies imply that an infinitesimal chiral transformation produces a specific, explicitly known change in Z, given by a polynomial of fourth order in the external vector and axial vector fields. (The anomalous contribution from the gluon operator tr $G_{\mu\nu}\tilde{G}^{\mu\nu}$ is accounted for by a change in the vacuum angle which is also treated as an external field.)

We construct the general solution of the Ward identities by power series expansion in the external momenta and in the masses of the three light quarks u, d and s. The leading contribution in this expansion is governed by classical field theory: the leading low energy representation of the generating functional $[O(p^2,\mathcal{M})]$ is given by the classical (unquantized) action of the nonlinear σ-model which is characterized by two constants F_0 and B_0. The main result of the paper is an explicit representation of the generating functional at first nonleading order in the low energy expansion $[O(p^4,p^2\mathcal{M},\mathcal{M}^2)]$. We show that at this order the Ward identities determine the Green's functions of the octet of vector and axial vector currents and of the nonet of scalar and pseudoscalar densities in terms of ten low energy coupling constants $L_1,\ldots, L_{10}$. We evaluate the corresponding quark mass expansion of the vacuum expectation values $\langle 0|\bar{u}u|0\rangle$, $\langle 0|\bar{d}d|0\rangle$, $\langle 0|\bar{s}s|0\rangle$ to order $\mathcal{M}$, the masses of the eight Goldstone bosons to order $\mathcal{M}^2$ and the decay constants F_π, F_K, F_η to order $\mathcal{M}$. (The expansion of the vacuum expectation values in powers of the quark masses involves an additional constant which specifies the conventions used in the subtraction of the perturbative infinities occurring in these quantities.)

Forming suitable combinations of physical low energy parameters the constants L_i may be eliminated. Examples are

$$\frac{m_d - m_u}{m_s - \hat{m}} \cdot \frac{2\hat{m}}{m_s + \hat{m}} = \frac{M_{K^0}^2 - M_{K^+}^2 - M_{\pi^0}^2 + M_{\pi^+}^2}{M_K^2 - M_\pi^2} \cdot \frac{M_\pi^2}{M_K^2} \left\{ 1 + O(m^2) + O\left(e^2 \frac{m_s - \hat{m}}{m_d - m_u}\right) \right\}$$

$$F_\eta = 1.02\, F_\pi (F_K / F_\pi)^{4/3} \qquad (15.1)$$

$$\varepsilon_d = \frac{1}{43.5} (\varepsilon_s + 0.13)$$

where ε_d, ε_s stand for the vacuum asymmetries

$$\frac{\langle 0 | \bar{d} d | 0 \rangle}{\langle 0 | \bar{u} u | 0 \rangle} = 1 - \varepsilon_d \quad ; \quad \frac{\langle 0 | \bar{s} s | 0 \rangle}{\langle 0 | \bar{u} u | 0 \rangle} = 1 - \varepsilon_s \qquad (15.2)$$

With the rather accurate value of $(m_d - m_u) : (m_s - \hat{m})$ obtained from isospin breaking in the baryon spectrum and from ρ-ω mixing the above relation between quark mass ratios and meson mass ratios implies

$$\frac{m_s}{\hat{m}} = 25.7 \pm 2.6 \qquad (15.3)$$

which is in good agreement with the value 25.0 $\pm$ 2.5 obtained from the ratios $M_K^2 : M_\pi^2$ and $M_\eta^2 : M_\pi^2$ in ref. I.

Analyzing the low energy representation for observables which are sensitive to the low energy coupling constants (such as $F_K : F_\pi$, determined by L_5) we show that the values of seven of the ten low energy constants may be extracted from measured quantities: the values are given in the table. To estimate the remaining three low energy constants (L_1, L_4 and L_6) we study the low energy expansion in the large N_c limit. In this limit the η' plays a special role. We show that the large N_c predictions for the constant L_7, which in this limit is dominated by η'-exchange, is consistent with a direct measurement of this quantity. (In this connection we point out that it is misleading to evaluate the $\eta\eta'$ mixing angle on the basis of the small deviation between the mass of the η and the mass predicted on the basis of the Gell-Mann-Okubo formula: other effects of order M^2 contribute to the η mass.) We show that in the large N_c limit the constants $2L_1 - L_2$,

L_4 and L_6 are suppressed and make an attempt to quantify this suppression by giving an estimate of the range in these quantities which is permitted by the Zweig rule.

The vacuum expectation values and the decay constants and masses of the Goldstone bosons characterize the chiral properties of the ground state. In these basic quantities the corrections of relative order m_s turn out to be small, thus confirming that the mass of the strange quark is small in comparison with the intrinsic scale of the theory. The infrared singularities of chiral perturbation theory turn out not to amplify the flavour asymmetries generated by the quark mass term (such as $\langle 0|\bar{u}u|0\rangle \neq \langle 0|\bar{s}s|0\rangle$, $F_K \neq F_\pi$) in a significant manner: these quantities are infrared stable. This is illustrated in the figures where we plot the masses, decay constants and vacuum expectation values as functions of the quark masses (at fixed ratios $m_u : m_d : m_s$). The infrared singularities manifest themselves in the curvature of these functions.

As will be shown elsewhere it is by no means the case that all low energy observables are infrared stable. The amplitude of the decay $\eta \to 3\pi$ e.g. is strongly affected by the corrections of relative order m_s.

In the present paper we have shown that the coupling constants of the effective Lagrangian can be pinned down in terms of infrared stable observables. The Green's functions of the theory may now be worked out to first nonleading order in a parameter free manner.

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We profited from illuminating discussions with P. Minkowski and with G. Veneziano.

Appendix

The kernels $M_{\mu\nu}(z)$, $K_\mu(z)$ and $J(z)$ defined in eq. (8.6) are linear combinations of the functions $-i\Delta_P(z)\,\Delta_Q(z)$, $-i\partial_\mu\Delta_P(z)\,\Delta_Q(z)$ and $-i\partial_\mu\partial_\nu\Delta_P(z)\Delta_Q(z)$. We consider therefore the Fourier transforms ($s = p^2$)

$$
\begin{aligned}
-i\int d^dz\, e^{ipz}\,\Delta_P(z)\Delta_Q(z) &= J(s)\\
-i\int d^dz\, e^{ipz}\,\partial_\mu\Delta_P(z)\Delta_Q(z) &= -ip_\mu J_1(s)\\
-i\int d^dz\, e^{ipz}\,\partial_\mu\partial_\nu\Delta_P(z)\Delta_Q(z) &= -\left[p_\mu p_\nu J_2(s) + g_{\mu\nu}\, s\, J_3(s)\right]
\end{aligned}
\tag{A.1}
$$

which are given by the standard one loop integrals

$$
-i\int \frac{d^dq}{(2\pi)^d}(M_P^2-q^2)^{-1}(M_Q^2-(p-q)^2)^{-1}\,(1;\, q_\mu;\, q_\mu q_\nu) = (J;\, p_\mu J_1;\, p_\mu p_\nu J_2 + g_{\mu\nu}\, s\, J_3) \tag{A.2}
$$

The scalar functions J_1, J_2 and J_3 may be expressed in terms of $J(s)$:

$$
\begin{aligned}
J_1(s) &= \frac{1}{2s}\,\frac{1}{i}\left(\Delta_P(0)-\Delta_Q(0)\right) + \frac{1}{2s}(s+\Delta)\,J(s)\\
J_2(s) &= -\frac{1}{d-1}\,\frac{1}{s^2}\left(sI_1 - dI_2\right)\\
J_3(s) &= \frac{1}{d-1}\,\frac{1}{s^2}\left(sI_1 - I_2\right)
\end{aligned}
\tag{A.3}
$$

where $\Delta_P(0)$ is the Feynman propagator evaluated at $z = 0$,

$$
-i\Delta_P(0) = \frac{M_P^2}{16\pi^2}\,\Gamma\left(1-\frac{d}{2}\right)\left(\frac{M_P}{2\sqrt{\pi}}\right)^{d-4} \tag{A.4}
$$

and

$$I_1(s) = M_P^2 J(s) - \frac{1}{i}\Delta_Q(0)$$

$$I_2(s) = \frac{1}{4}\left\{ \frac{1}{i}\left(\Delta_P(0) - \Delta_Q(0)\right)(s+\Delta) - \frac{2s}{i}\Delta_Q(0) + (s+\Delta)^2 J(s)\right\}$$

$$\Delta = M_P^2 - M_Q^2 \qquad \text{(A.5)}$$

It suffices therefore to discuss the function J(s) - the kernels $M_{\mu\nu}$ and K_ν may then easily be obtained from eqs. ((8.6) and (A.3)-(A.5)).

For the following it is convenient to make use of the integral representation

$$J(s) = \pi^{d/2}(2\pi)^{-d}\,\Gamma(2-\tfrac{d}{2})\int_0^1 g(x;s)^{\frac{d-4}{2}}\,dx$$

$$g(x;s) = M_P^2 - sx(1-x) - \Delta\cdot x \qquad \text{(A.6)}$$

which follows in a standard manner from eq. (A.2).

J(s) develops a pole as d → 4. The quantity $\bar{J}(s)$ defined by

$$\bar{J}(s) = J(s) - J(0)$$

remains finite as d → 4:

$$\bar{J}(s) = -(16\pi^2)^{-1}\int_0^1 \ln\left[g(x;s)/g(x;0)\right]dx$$

$$= (32\pi^2)^{-1}\left\{2 + \frac{\Delta}{s}\ln\frac{M_Q^2}{M_P^2} - \frac{\Sigma}{\Delta}\ln\frac{M_Q^2}{M_P^2} - \frac{\nu}{s}\ln\frac{(s+\nu)^2-\Delta^2}{(s-\nu)^2-\Delta^2}\right\} + O(d-4) \qquad \text{(A.7)}$$

where

$$\Sigma = M_P^2 + M_Q^2 \tag{A.8}$$

$$\nu^2 = \left[s - (M_P + M_Q)^2\right]\left[s - (M_P - M_Q)^2\right]$$

The pole is contained in J(0):

$$J(0) = -2\lambda - 2k + O(d-4)$$

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1\right)\right\} \tag{A.9}$$

$$k = \frac{1}{32\pi^2}\left\{M_P^2 \ln\frac{M_P^2}{\mu^2} - M_Q^2 \ln\frac{M_Q^2}{\mu^2}\right\}\frac{1}{\Delta}$$

For convenience we also give the values of the derivatives of the function $\bar{J}(s)$ at s = 0, which are easily obtained from the integral representation (A.6)

$$\bar{J}'(0) = \frac{1}{32\pi^2}\left\{\frac{\Sigma}{\Delta^2} + 2\frac{M_P^2 M_Q^2}{\Delta^3}\ln\frac{M_Q^2}{M_P^2}\right\} \tag{A.10}$$

$$\bar{J}''(0) = \frac{1}{32\pi^2}\left\{\frac{2}{3\Delta^4}\left(3\Sigma^2 - 2\Delta^2\right) + 4\frac{M_P^2 M_Q^2}{\Delta^5}\Sigma\ln\frac{M_Q^2}{M_P^2}\right\}$$

If the two masses are equal, $M_P^2 = M_Q^2 = M^2$, the function $\bar{J}(s)$ and the constants $\bar{J}'(0)$, $\bar{J}''(0)$, k simplify to

$$\bar{J}(s) = \frac{1}{16\pi^2}\left\{\sigma \ln\frac{\sigma-1}{\sigma+1} + 2\right\}$$

$$\sigma = (1 - 4M^2/s)^{1/2} \tag{A.11}$$

$$\bar{J}'(0) = \frac{1}{96\pi^2}\frac{1}{M^2} \; ; \; \bar{J}''(0) = \frac{1}{480\pi^2}\frac{1}{M^4}$$

$$k = \frac{1}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} + 1\right)$$

Table: Values of the low energy coupling constants (running scale taken at $\mu = M_\eta$).

	Value	Source	Γ_i
L_1^r	$(0.9 \pm 0.3) \cdot 10^{-3}$	$\pi\pi$ D-waves, Zweig rule	$\frac{3}{32}$
L_2^r	$(1.7 \pm 0.7) \cdot 10^{-3}$	$\pi\pi$ D-waves	$\frac{3}{16}$
L_3	$(- 4.4 \pm 2.5) \cdot 10^{-3}$	$\pi\pi$ D-waves, Zweig rule	0
L_4^r	$(0 \pm 0.5) \cdot 10^{-3}$	Zweig rule	$\frac{1}{8}$
L_5^r	$(2.2 \pm 0.5) \cdot 10^{-3}$	$F_K : F_\pi$	$\frac{3}{8}$
L_6^r	$(0 \pm 0.3) \cdot 10^{-3}$	Zweig rule	$\frac{11}{144}$
L_7	$(- 0.4 \pm 0.15) \cdot 10^{-3}$	Gell-Mann-Okubo, L_5, L_8	0
L_8^r	$(1.1 \pm 0.3) \cdot 10^{-3}$	$K^0 - K^+$, R, L_5	$\frac{5}{48}$
L_9^r	$(7.4 \pm 0.7) \cdot 10^{-3}$	$<r^2>^\pi_{e.m.}$	$\frac{1}{4}$
L_{10}^r	$(- 6.0 \pm 0.7) \cdot 10^{-3}$	$\pi \to e\nu\gamma$	$- \frac{1}{4}$

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Figure captions

Fig. 1: The masses of the pseudoscalar octet as a function of m_s at fixed ratios $m_u : m_d : m_s$ (calculated from eq. (10.7), using the central values of the low energy constants quoted in the table). The dashed line gives the value of M_η^2 without the contribution from $\eta\eta'$ mixing ($L_7 = 0$). Note that the curves are independent of the absolute value used for m_s^{phys}.

Fig. 2: The decay constants F_π, F_K and F_η as a function of m_s at fixed ratios $m_u : m_d : m_s$.

Fig. 3: The order parameters $<0|\bar{u}u|0>$ and $<0|\bar{s}s|0>$ as a function of m_s at fixed ratios $m_u : m_d : m_s$ (compare Novikov et al. 1981). These quantities depend on the high energy constant H_2 which specifies the subtraction prescription in $<0|\bar{q}q|0>$. We show the results for two values of this constant, chosen such that at the physical value of m_s the asymmetry $\varepsilon_s = 1 - <0|\bar{s}s|0>/<0|\bar{u}u|0>$ becomes $\varepsilon_s = +\ 0.3$ and $-\ 0.1$ respectively. (The isospin asymmetry $<0|\bar{d}d|0> - <0|\bar{u}u|0>$ and the sensitivity of $<0|\bar{u}u|0>$ to ε_s are not visible on this scale.) For comparison we also plot the large m prediction of Shifman et al. (1979), with $G = <0|\frac{\alpha}{\pi} G^a_{\mu\nu} G^{\mu\nu a}|0> = 0.012\ \mathrm{GeV}^4$.

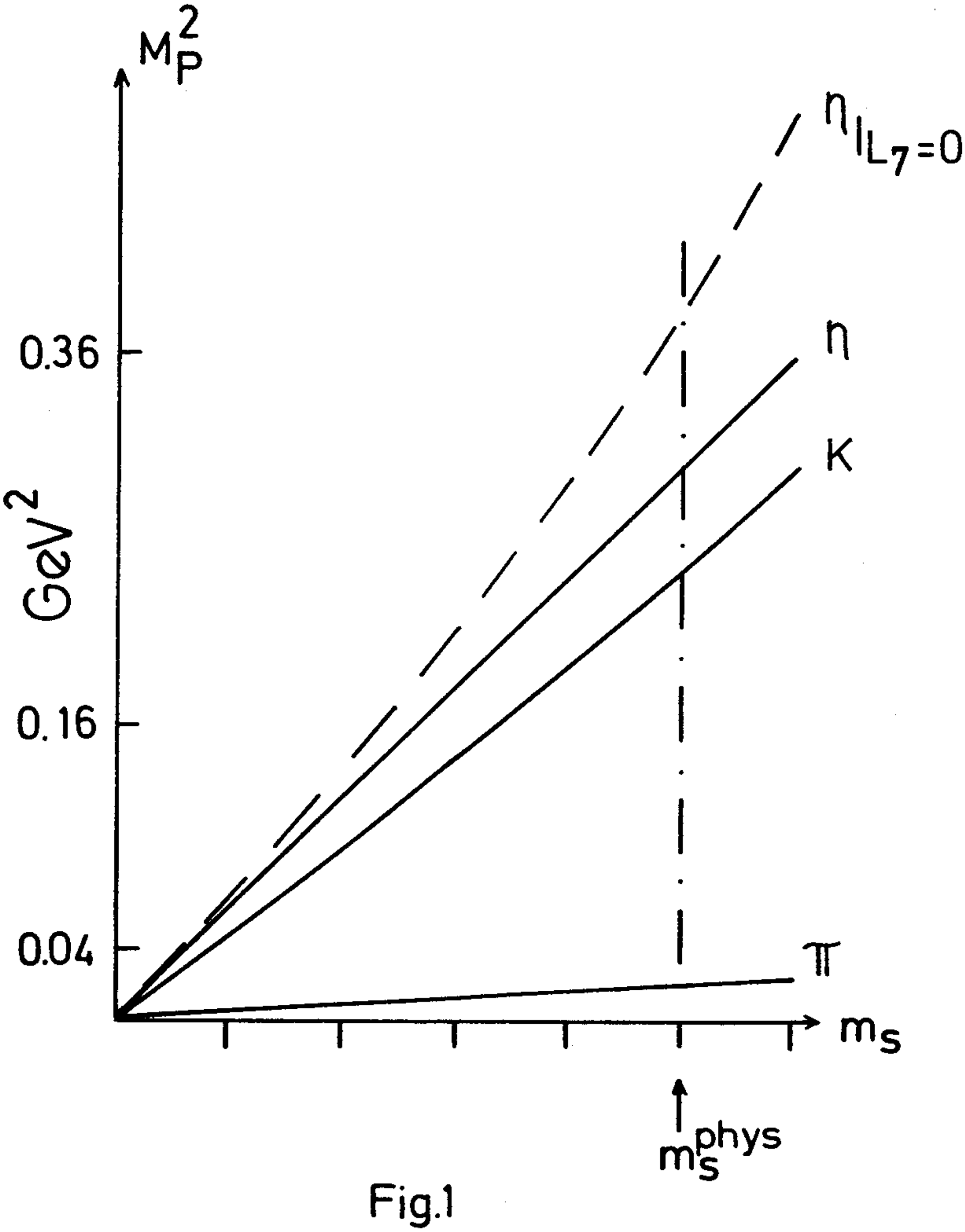

Fig.1

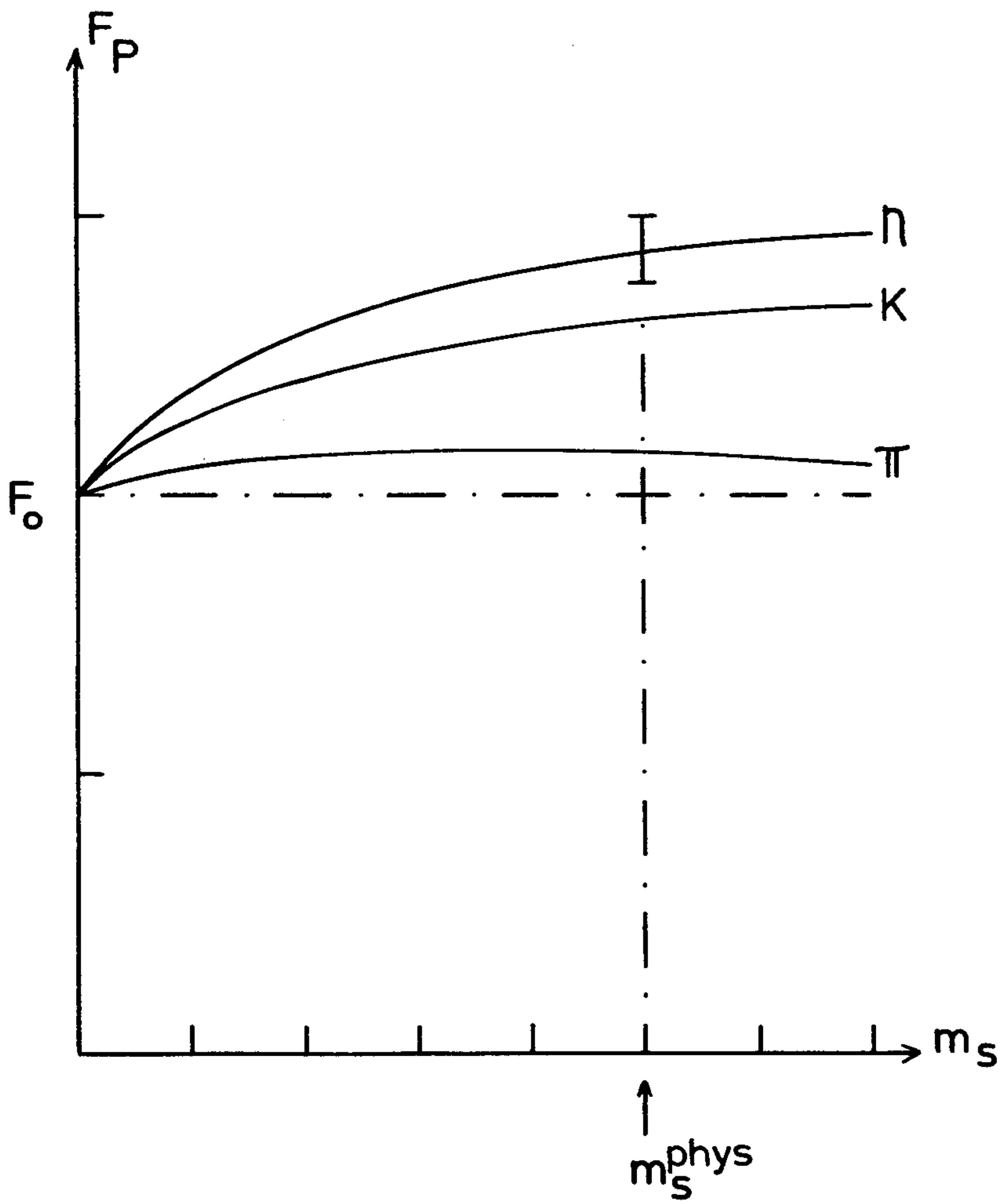

Fig.2

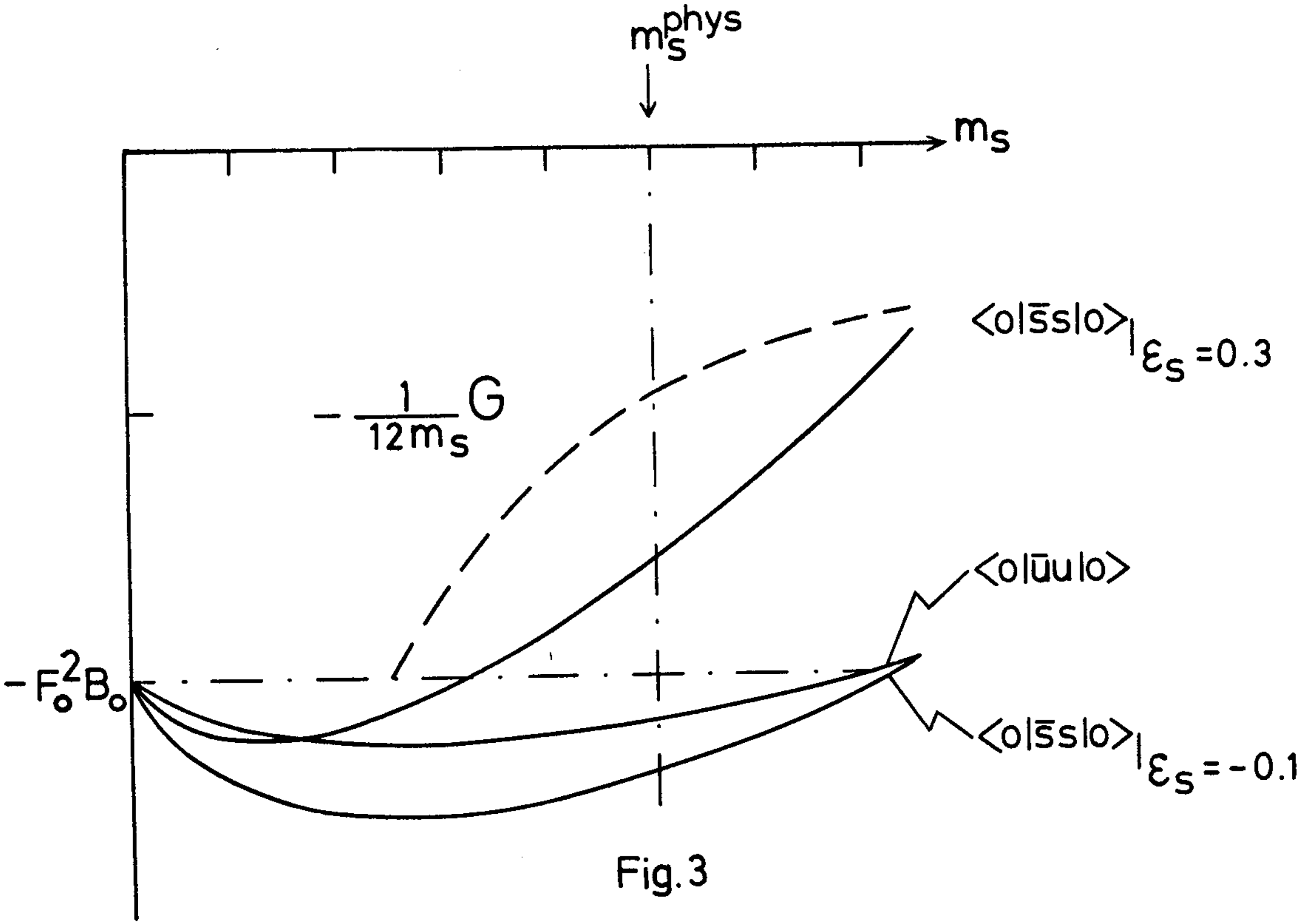
m_s^{phys}
m_s
$-\frac{1}{12m_s}G$
$\langle o|\bar{s}s|o\rangle|_{\varepsilon_s=0.3}$
$\langle o|\bar{u}u|o\rangle$
$\langle o|\bar{s}s|o\rangle|_{\varepsilon_s=-0.1}$
$-F_o^2B_o$

Fig. 3